(a) The vector spaces R^2 and P_2 are isomorphic.

(b) If the kernel of a linear transformation $T: P_3 \to P_3$ is $\{\mathbf{0}\}$, then T is an isomorphism.

(c) Every linear transformation from M_{33} to P_9 is an isomorphism.

(d) There is a subspace of M_{23} that is isomorphic to R^4.

(e) Isomorphic finite-dimensional vector spaces must have the same number of basis vectors.

(f) R^n is isomorphic to a subspace of R^{n+1}.

8.4 Matrices for General Linear Transformations

In this section we will show that a general linear transformation from any n-dimensional vector space V to any m-dimensional vector space W can be performed using an appropriate matrix transformation from R^n to R^m. This idea is used in computer computations since computers are well suited for performing matrix computations.

Matrices of Linear Transformations

Suppose that V is an n-dimensional vector space, that W is an m-dimensional vector space, and that $T: V \to W$ is a linear transformation. Suppose further that B is a basis for V, that B' is a basis for W, and that for each vector $\mathbf{x}$ in V, the coordinate matrices for $\mathbf{x}$ and $T(\mathbf{x})$ are $[\mathbf{x}]_B$ and $[T(\mathbf{x})]_{B'}$, respectively (Figure 8.4.1).

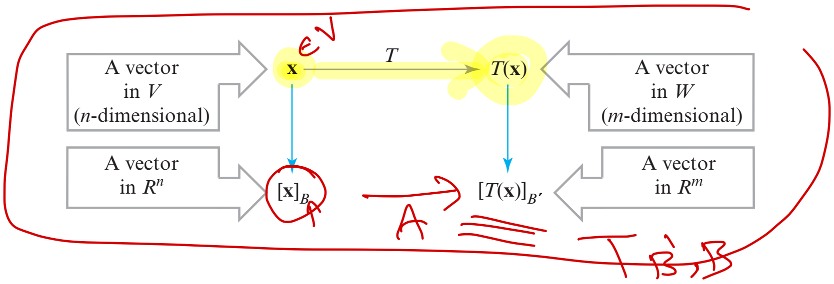


Figure 8.4.1

It will be our goal to find an $m \times n$ matrix A such that multiplication by A maps the vector $[\mathbf{x}]_B$ into the vector $[T(\mathbf{x})]_{B'}$ for each $\mathbf{x}$ in V (Figure 8.4.2a). If we can do so, then, as illustrated in Figure 8.4.2b, we will be able to execute the linear transformation T by using matrix multiplication and the following *indirect* procedure:

Finding $T(\mathbf{x})$ Indirectly

Step 1. Compute the coordinate vector $[\mathbf{x}]_B$.

Step 2. Multiply $[\mathbf{x}]_B$ on the left by A to produce $[T(\mathbf{x})]_{B'}$.

Step 3. Reconstruct $T(\mathbf{x})$ from its coordinate vector $[T(\mathbf{x})]_{B'}$.

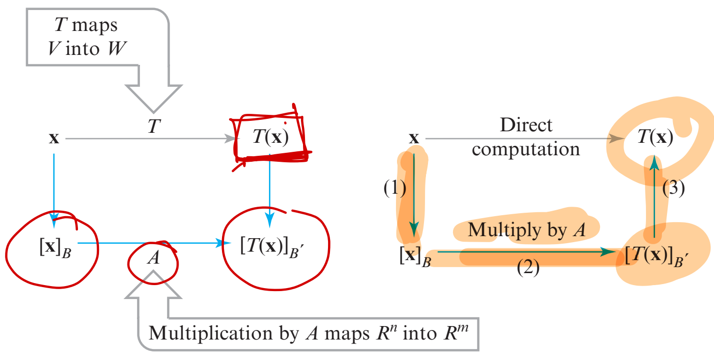


Figure 8.4.2 (a) (b)

The key to executing this plan is to find an $m \times n$ matrix A with the property that

$$A[\mathbf{x}]_B = [T(\mathbf{x})]_{B'} \tag{1}$$

For this purpose, let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a basis for the n-dimensional space V and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ a basis for the m-dimensional space W. Since Equation (1) must hold for all vectors in V, it must hold, in particular, for the basis vectors in B; that is,

$$A[\mathbf{u}_1]_B = [T(\mathbf{u}_1)]_{B'}, \quad A[\mathbf{u}_2]_B = [T(\mathbf{u}_2)]_{B'}, \dots, \quad A[\mathbf{u}_n]_B = [T(\mathbf{u}_n)]_{B'} \tag{2}$$

But

$$[\mathbf{u}_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad [\mathbf{u}_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \quad [\mathbf{u}_n]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

so

$$A[\mathbf{u}_1]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$A[\mathbf{u}_2]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$A[\mathbf{u}_n]_B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Substituting these results into (2) yields

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = [T(\mathbf{u}_1)]_{B'}, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} = [T(\mathbf{u}_2)]_{B'}, \dots, \quad \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = [T(\mathbf{u}_n)]_{B'}$$

which shows that the successive columns of A are the coordinate vectors of

$$T(\mathbf{u}_1), T(\mathbf{u}_2), \dots, T(\mathbf{u}_n)$$

with respect to the basis B'. Thus, the matrix A that completes the link in Figure 8.4.2a is

$$A = \big[[T(\mathbf{u}_1)]_{B'} \mid [T(\mathbf{u}_2)]_{B'} \mid \cdots \mid [T(\mathbf{u}_n)]_{B'}\big] \tag{3}$$

We will call this the ***matrix for T relative to the bases B and B′*** and will denote it by the symbol $[T]_{B',B}$. Using this notation, Formula (3) can be written as

$$[T]_{B',B} = \big[[T(\mathbf{u}_1)]_{B'} \mid [T(\mathbf{u}_2)]_{B'} \mid \cdots \mid [T(\mathbf{u}_n)]_{B'}\big] \tag{4}$$

A

$u_i \in$ Basis of V

and from (1), this matrix has the property

$$[T]_{B',B}[\mathbf{x}]_B = [T(\mathbf{x})]_{B'} \tag{5}$$

We leave it as an exercise to show that in the special case where $T_C\colon R^n \to R^m$ is multiplication by C, and where B and B' are the *standard bases* for R^n and R^m, respectively, then

$$[T_C]_{B',B} = C \tag{6}$$

$$[T]_{B',B}$$

Basis for the image space	Basis for the domain

▲ Figure 8.4.3

$$[T]_{B',B}[\mathbf{x}]_B = [T(\mathbf{x})]_{B'}$$

Cancellation

▲ Figure 8.4.4

Remark Observe that in the notation $[T]_{B',B}$ the right subscript is a basis for the domain of T, and the left subscript is a basis for the image space of T (Figure 8.4.3). Moreover, observe how the subscript B seems to "cancel out" in Formula (5) (Figure 8.4.4).

▶ EXAMPLE 1 Matrix for a Linear Transformation

Let $T: P_1 \to P_2$ be the linear transformation defined by

V W

$$T(p(x)) = xp(x)$$

$\in P_1$ $\in P_2$

Find the matrix for T with respect to the standard bases

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} \quad \text{and} \quad B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

P_1 $\quad\to P_2$

$B=\{1, x\}$

$B'=\{1, x, x^2\}$

where

$$\mathbf{u}_1 = 1, \quad \mathbf{u}_2 = x; \qquad \mathbf{v}_1 = 1, \quad \mathbf{v}_2 = x, \quad \mathbf{v}_3 = x^2$$

Solution From the given formula for T we obtain

1

$$T(\mathbf{u}_1) = T(1) = (x)(1) = x$$
$$T(\mathbf{u}_2) = T(x) = (x)(x) = x^2$$

$x = C_1(1) + C_2(x) + C_3(x^2)$

$C_1 = 0$, $C_2 = 1$, $C_3 = 0$

By inspection, the coordinate vectors for $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ relative to B' are

2

$$[T(\mathbf{u}_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(\mathbf{u}_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$x^2 = C_1(1) + C_2(x) + C_3(x^2)$

$C_1 = 0$, $C_2 = 0$, $C_3 = 1$

Thus, the matrix for T with respect to B and B' is

3

$$[T]_{B',B} = \big[[T(\mathbf{u}_1)]_{B'} \mid [T(\mathbf{u}_2)]_{B'}\big] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

▶ EXAMPLE 2 The Three-Step Procedure

Let $T: P_1 \to P_2$ be the linear transformation in Example 1, and use the three-step procedure described in the following figure to perform the computation

$$T(a + bx) = x(a + bx) = ax + bx^2$$

given

$\in P_1$

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "\text{Direct computation}"] \arrow[d, "(1)"'] & T(\mathbf{x}) \\
{[\mathbf{x}]_B} \arrow[r, "\text{Multiply by } {[T]_{B',B}}", "(2)"'] & {[T(\mathbf{x})]_{B'}} \arrow[u, "(3)"']
\end{tikzcd}$$

$a_0 + a_1 x$

Solution

$a+bx = C_1(1)+C_2(x) \Rightarrow C_1 = a$, $C_2 = b$

Step 1. The coordinate matrix for $\mathbf{x} = a + bx$ relative to the basis $B = \{1, x\}$ is

$$[\mathbf{x}]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Step 2. Multiplying $[\mathbf{x}]_B$ by the matrix $[T]_{B',B}$ found in Example 1 we obtain

$$[T]_{B',B}[\mathbf{x}]_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} = [T(\mathbf{x})]_{B'}$$

$\in P_2$ $\quad a_0 + a_1x + a_2x^2$

$a_0 = 0$, $a_1 = a$, $a_2 = b$

> Although Example 2 is simple, the procedure that it illustrates is applicable to problems of great complexity.

Step 3. Reconstructing $T(\mathbf{x}) = T(a + bx)$ from $[T(\mathbf{x})]_{B'}$ we obtain

$$T(a + bx) = 0 + ax + bx^2 = ax + bx^2$$

$T(a+bx) = ax + bx^2$

▶ EXAMPLE 3 Matrix for a Linear Transformation

Let $T: R^2 \to R^3$ be the linear transformation defined by

V W

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the matrix for the transformation T with respect to the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2 and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution From the formula for T,

$T\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5(3)+13(1) \\ -7(3)+16(1) \end{bmatrix}$

$$T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad T(\mathbf{u}_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

$T\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

Expressing these vectors as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we obtain (verify)

$$T(\mathbf{u}_1) = \mathbf{v}_1 - 2\mathbf{v}_3, \quad T(\mathbf{u}_2) = 3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$$

$\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + C_3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$C_1 - C_2 = 1 \Rightarrow C_1 = 1 + C_2$

$2C_2 + C_3 = -2 \Rightarrow C_3 = -2 - 2C_2$

$-C_1 + 2C_2 + 2C_3 = -5$

$-1 - C_2 + 2C_2 + 2(-2 - 2C_2) = -5$

$-1 + C_2 - 4 - 4C_2 = -5$

$\cancel{-5} - 3C_2 = \cancel{-5}$

$\boxed{C_2 = 0}$

$\boxed{C_1 = 1}$

$\boxed{C_3 = -2}$

$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$C_1 - C_2 = 2$

$2C_2 + C_3 = 1$

$-C_1 + 2C_2 + 2C_3 = -3$

Thus,

$$[T(\mathbf{u}_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [T(\mathbf{u}_2)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

so

$$[T]_{B',B} = \big[[T(\mathbf{u}_1)]_{B'} \mid [T(\mathbf{u}_2)]_{B'}\big] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix} \blacktriangleleft$$

$\boxed{C_1 = 3, \ C_2 = 1, \ C_3 = -1}$

Remark Example 3 illustrates that a fixed linear transformation generally has multiple representations, each depending on the bases chosen. In this case the matrices

$$[T] = \begin{bmatrix} 0 & 1 \\ -5 & 13 \\ -7 & 16 \end{bmatrix} \quad \text{and} \quad [T]_{B',B} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

both represent the transformation T, the first relative to the standard bases for R^2 and R^3, the second relative to the bases B and B' stated in the example.

$T: V \to V$

Matrices of Linear Operators

In the special case where $V = W$ (so that $T: V \to V$ is a linear operator), it is usual to take $B = B'$ when constructing a matrix for T. In this case the resulting matrix is called the ***matrix for T relative to the basis B*** and is usually denoted by $[T]_B$ rather than $[T]_{B,B}$. If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, then Formulas (4) and (5) become

$$[T]_B = \big[[T(\mathbf{u}_1)]_B \mid [T(\mathbf{u}_2)]_B \mid \cdots \mid [T(\mathbf{u}_n)]_B\big] \tag{7}$$

$$[T]_B[\mathbf{x}]_B = [T(\mathbf{x})]_B \tag{8}$$

Phrased informally, Formulas (7) and (8) state that *the matrix for T, when multiplied by the coordinate vector for* $\mathbf{x}$, *produces the coordinate vector for* $T(\mathbf{x})$.

In the special case where $T: R^n \to R^n$ is a matrix operator, say multiplication by A, and B is the standard basis for R^n, then Formula (7) simplifies to

$$[T]_B = A \tag{9}$$

Matrices of Identity Operators

Recall that the identity operator $I: V \to V$ maps every vector in V into itself, that is, $I(\mathbf{x}) = \mathbf{x}$ for every vector $\mathbf{x}$ in V. The following example shows that if V is n-dimensional, then the matrix for I relative to *any* basis B for V is the $n \times n$ identity matrix.

EXAMPLE 4 Matrices of Identity Operators

If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for a finite-dimensional vector space V, and if $I: V \to V$ is the identity operator on V, then

$$I(\mathbf{u}_1) = \mathbf{u}_1, \quad I(\mathbf{u}_2) = \mathbf{u}_2, \ldots, \quad I(\mathbf{u}_n) = \mathbf{u}_n$$

Therefore,

$$[I]_B = \begin{array}{c} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \uparrow \\ [I(\mathbf{u}_1)]_B & [I(\mathbf{u}_2)]_B & & [I(\mathbf{u}_n)]_B \end{matrix} \end{array} = I$$

EXAMPLE 5 Linear Operator on P_2

Let $T: P_2 \to P_2$ be the linear operator defined by

$$T(p(x)) = p(3x - 5)$$

~~that is,~~ $\cancel{T(c_0 + c_1x + c_2x^2) = c_0 + c_1(3x - 5)} + c_2(3x - 5)^2$.

(a) Find $[T]_B$ relative to the basis $B = \{1, x, x^2\}$.

(b) Use the indirect procedure to compute $T(1 + 2x + 3x^2)$.

(c) Check the result in (b) by computing $T(1 + 2x + 3x^2)$ directly.

$T(x) = 3x - 5$

$T(2x-1) = 2(3x-5) - 1$

$= 6x - 10 - 1$

$= 6x - 11$

$1 = c_1(1) + c_2(x) + c_3(x^2)$

$u_1 = 1, \; u_2 = x, \; u_3 = x^2$

Solution (a) From the formula for T,

$3x - 5 = c_1(1) + c_2(x) + c_3(x^2)$

$$T(1) = 1, \quad T(x) = 3x - 5, \quad T(x^2) = (3x - 5)^2 = 9x^2 - 30x + 25$$

$9x^2 - 30x + 25 = c_1(1) + c_2(x) + c_3(x^2)$

so

$$[T(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_B = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, \quad [T(x^2)]_B = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}$$

Thus,

$$[T]_B = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$

Solution (b)

Step 1. The coordinate matrix for $\mathbf{p} = 1 + 2x + 3x^2$ relative to the basis $B = \{1, x, x^2\}$ is

$$[\mathbf{p}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$1 + 2x + 3x^2 = c_1(1) + c_2(x) + c_3(x^2)$

Step 2. Multiplying $[\mathbf{p}]_B$ by the matrix $[T]_B$ found in part (a) we obtain

$$[T]_B[\mathbf{p}]_B = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 66 \\ -84 \\ 27 \end{bmatrix} = [T(\mathbf{p})]_B$$

$a_0 + a_1 x + a_2 x^2$

$66 - 84x + 27x^2$

Step 3. Reconstructing $T(\mathbf{p}) = T(1 + 2x + 3x^2)$ from $[T(\mathbf{p})]_B$ we obtain

$$T(1 + 2x + 3x^2) = 66 - 84x + 27x^2$$

Solution (c) By direct computation,

$$\begin{aligned} T(1 + 2x + 3x^2) &= 1 + 2(3x - 5) + 3(3x - 5)^2 \\ &= 1 + 6x - 10 + 27x^2 - 90x + 75 \\ &= 66 - 84x + 27x^2 \end{aligned}$$

which agrees with the result in (b). ◀

Matrices of Compositions and Inverse Transformations

We will conclude this section by mentioning two theorems without proof that are generalizations of Formulas (4) and (9) of Section 4.10.

THEOREM 8.4.1 *If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations, and if B, B'', and B' are bases for U, V, and W, respectively, then*

$$[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''}[T_1]_{B'',B} \tag{10}$$

THEOREM 8.4.2 *If $T: V \to V$ is a linear operator, and if B is a basis for V, then the following are equivalent.*

(a) T is one-to-one.

(b) $[T]_B$ is invertible.

Moreover, when these equivalent conditions hold,

$$[T^{-1}]_B = [T]_B^{-1} \tag{11}$$

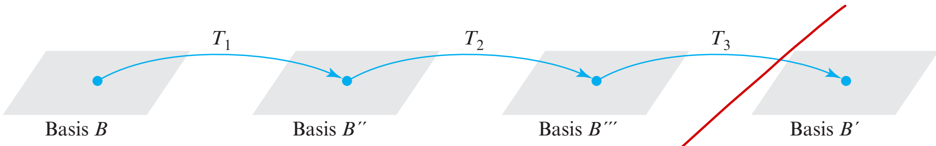
Remark In (10), observe how the interior subscript B'' (the basis for the intermediate space V) seems to "cancel out," leaving only the bases for the domain and image space of the composition as subscripts (Figure 8.4.5). This "cancellation" of interior subscripts suggests the following extension of Formula (10) to compositions of three linear transformations (Figure 8.4.6):

$$[T_3 \circ T_2 \circ T_1]_{B',B} = [T_3]_{B',B'''}[T_2]_{B''',B''}[T_1]_{B'',B} \tag{12}$$

$$[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''}[T_1]_{B'',B}$$

Cancellation

▲ Figure 8.4.5



▲ Figure 8.4.6

The following example illustrates Theorem 8.4.1.

▶ EXAMPLE 6 Composition

Let $T_1: P_1 \to P_2$ be the linear transformation defined by

$$T_1(p(x)) = xp(x)$$

and let $T_2: P_2 \to P_2$ be the linear operator defined by

$$T_2(p(x)) = p(3x-5)$$

Then the composition $(T_2 \circ T_1): P_1 \to P_2$ is given by

$$(T_2 \circ T_1)(p(x)) = T_2(T_1(p(x))) = T_2(xp(x)) = (3x-5)p(3x-5)$$

Thus, if $p(x) = c_0 + c_1 x$, then

$$\begin{aligned}(T_2 \circ T_1)(c_0 + c_1 x) &= (3x-5)\big(c_0 + c_1(3x-5)\big)\\ &= c_0(3x-5) + c_1(3x-5)^2 \end{aligned} \tag{13}$$

In this example, P_1 plays the role of U in Theorem 8.4.1, and P_2 plays the roles of both V and W; thus we can take $B' = B''$ in (10) so that the formula simplifies to

$$[T_2 \circ T_1]_{B',B} = [T_2]_{B'}[T_1]_{B',B} \tag{14}$$

Let us choose $B = \{1, x\}$ to be the basis for P_1 and choose $B' = \{1, x, x^2\}$ to be the basis for P_2. We showed in Examples 1 and 5 that

$$[T_1]_{B',B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_2]_{B'} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$$

Thus, it follows from (14) that

$$[T_2 \circ T_1]_{B',B} = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 25 \\ 3 & -30 \\ 0 & 9 \end{bmatrix} \tag{15}$$

As a check, we will calculate $[T_2 \circ T_1]_{B',B}$ directly from Formula (4). Since $B = \{1, x\}$, it follows from Formula (4) with $\mathbf{u}_1 = 1$ and $\mathbf{u}_2 = x$ that

$$[T_2 \circ T_1]_{B',B} = \big[[(T_2 \circ T_1)(1)]_{B'} \mid [(T_2 \circ T_1)(x)]_{B'}\big] \tag{16}$$

Using (13) yields

$$(T_2 \circ T_1)(1) = 3x - 5 \quad \text{and} \quad (T_2 \circ T_1)(x) = (3x-5)^2 = 9x^2 - 30x + 25$$

From this and the fact that $B' = \{1, x, x^2\}$, it follows that

$$[(T_2 \circ T_1)(1)]_{B'} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad [(T_2 \circ T_1)(x)]_{B'} = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}$$

Substituting in (16) yields

$$[T_2 \circ T_1]_{B',B} = \begin{bmatrix} -5 & 25 \\ 3 & -30 \\ 0 & 9 \end{bmatrix}$$

which agrees with (15). ◀

Exercise Set 8.4

1. Let $T: P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$.
   (a) Find the matrix for T relative to the standard bases
   $$B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \quad \text{and} \quad B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$
   where
   $$\begin{aligned} &\mathbf{u}_1 = 1, \quad \mathbf{u}_2 = x, \quad \mathbf{u}_3 = x^2 \\ &\mathbf{v}_1 = 1, \quad \mathbf{v}_2 = x, \quad \mathbf{v}_3 = x^2, \quad \mathbf{v}_4 = x^3 \end{aligned}$$
   (b) Verify that the matrix $[T]_{B',B}$ obtained in part (a) satisfies Formula (5) for every vector $\mathbf{x} = c_0 + c_1x + c_2x^2$ in P_2.

2. Let $T: P_2 \to P_1$ be the linear transformation defined by
   $$T(a_0 + a_1x + a_2x^2) = (a_0 + a_1) - (2a_1 + 3a_2)x$$
   (a) Find the matrix for T relative to the standard bases $B = \{1, x, x^2\}$ and $B' = \{1, x\}$ for P_2 and P_1.
   (b) Verify that the matrix $[T]_{B',B}$ obtained in part (a) satisfies Formula (5) for every vector $\mathbf{x} = c_0 + c_1x + c_2x^2$ in P_2.

3. Let $T: P_2 \to P_2$ be the linear operator defined by
   $$T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x - 1) + a_2(x - 1)^2$$
   (a) Find the matrix for T relative to the standard basis $B = \{1, x, x^2\}$ for P_2.
   (b) Verify that the matrix $[T]_B$ obtained in part (a) satisfies Formula (8) for every vector $\mathbf{x} = a_0 + a_1x + a_2x^2$ in P_2.

4. Let $T: R^2 \to R^2$ be the linear operator defined by
   $$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$
   and let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ be the basis for which
   $$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
   (a) Find $[T]_B$.
   (b) Verify that Formula (8) holds for every vector $\mathbf{x}$ in R^2.

5. Let $T: R^2 \to R^3$ be defined by
   $$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{bmatrix}$$
   (a) Find the matrix $[T]_{B',B}$ relative to the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where
   $$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$
   $$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$
   (b) Verify that Formula (5) holds for every vector in R^2.

6. Let $T: R^3 \to R^3$ be the linear operator defined by
   $$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_1, x_1 - x_3)$$
   (a) Find the matrix for T with respect to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where
   $$\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (0, 1, 1), \quad \mathbf{v}_3 = (1, 1, 0)$$
   (b) Verify that Formula (8) holds for every vector $\mathbf{x} = (x_1, x_2, x_3)$ in R^3.
   (c) Is T one-to-one? If so, find the matrix of T^{-1} with respect to the basis B.

7. Let $T: P_2 \to P_2$ be the linear operator defined by $T(p(x)) = p(2x + 1)$, that is,
   $$T(c_0 + c_1x + c_2x^2) = c_0 + c_1(2x + 1) + c_2(2x + 1)^2$$
   (a) Find $[T]_B$ with respect to the basis $B = \{1, x, x^2\}$.
   (b) Use the three-step procedure illustrated in Example 2 to compute $T(2 - 3x + 4x^2)$.
   (c) Check the result obtained in part (b) by computing $T(2 - 3x + 4x^2)$ directly.

8. Let $T: P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x - 3)$, that is,
   $$T(c_0 + c_1x + c_2x^2) = x\big(c_0 + c_1(x - 3) + c_2(x - 3)^2\big)$$
   (a) Find $[T]_{B',B}$ relative to the bases $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$.
   (b) Use the three-step procedure illustrated in Example 2 to compute $T(1 + x - x^2)$.
   (c) Check the result obtained in part (b) by computing $T(1 + x - x^2)$ directly.

9. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ be the matrix for $T: R^2 \to R^2$ relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$.
   (a) Find $[T(\mathbf{v}_1)]_B$ and $[T(\mathbf{v}_2)]_B$.
   (b) Find $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$.
   (c) Find a formula for $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.
   (d) Use the formula obtained in (c) to compute $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

10. Let $A = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 1 & 6 & 2 & 1 \\ -3 & 0 & 7 & 1 \end{bmatrix}$ be the matrix for $T: R^4 \to R^3$ relative to the bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and

$B' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 9 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 8 \\ 8 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -7 \\ 8 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} -6 \\ 9 \\ 1 \end{bmatrix}$$

(a) Find $[T(\mathbf{v}_1)]_{B'}$, $[T(\mathbf{v}_2)]_{B'}$, $[T(\mathbf{v}_3)]_{B'}$, and $[T(\mathbf{v}_4)]_{B'}$.

(b) Find $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, $T(\mathbf{v}_3)$, and $T(\mathbf{v}_4)$.

(c) Find a formula for $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right)$.

(d) Use the formula obtained in (c) to compute $T\left(\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}\right)$.

11. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$ be the matrix for $T: P_2 \to P_2$ with respect to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = 3x + 3x^2$, $\mathbf{v}_2 = -1 + 3x + 2x^2$, $\mathbf{v}_3 = 3 + 7x + 2x^2$.

(a) Find $[T(\mathbf{v}_1)]_B$, $[T(\mathbf{v}_2)]_B$, and $[T(\mathbf{v}_3)]_B$.

(b) Find $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $T(\mathbf{v}_3)$.

(c) Find a formula for $T(a_0 + a_1x + a_2x^2)$.

(d) Use the formula obtained in (c) to compute $T(1 + x^2)$.

12. Let $T_1: P_1 \to P_2$ be the linear transformation defined by

$$T_1(p(x)) = xp(x)$$

and let $T_2: P_2 \to P_2$ be the linear operator defined by

$$T_2(p(x)) = p(2x + 1)$$

Let $B = \{1, x\}$ and $B' = \{1, x, x^2\}$ be the standard bases for P_1 and P_2.

(a) Find $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B'}$, and $[T_1]_{B',B}$.

(b) State a formula relating the matrices in part (a).

(c) Verify that the matrices in part (a) satisfy the formula you stated in part (b).

13. Let $T_1: P_1 \to P_2$ be the linear transformation defined by

$$T_1(c_0 + c_1x) = 2c_0 - 3c_1x$$

and let $T_2: P_2 \to P_3$ be the linear transformation defined by

$$T_2(c_0 + c_1x + c_2x^2) = 3c_0x + 3c_1x^2 + 3c_2x^3$$

Let $B = \{1, x\}$, $B'' = \{1, x, x^2\}$, and $B' = \{1, x, x^2, x^3\}$.

(a) Find $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B',B''}$, and $[T_1]_{B'',B}$.

(b) State a formula relating the matrices in part (a).

(c) Verify that the matrices in part (a) satisfy the formula you stated in part (b).

14. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis for a vector space V. Find the matrix with respect to B for the linear operator $T: V \to V$ defined by $T(\mathbf{v}_1) = \mathbf{v}_2$, $T(\mathbf{v}_2) = \mathbf{v}_3$, $T(\mathbf{v}_3) = \mathbf{v}_4$, $T(\mathbf{v}_4) = \mathbf{v}_1$.

15. Let $T: P_2 \to M_{22}$ be the linear transformation defined by

$$T(\mathbf{p}) = \begin{bmatrix} p(0) & p(1) \\ p(-1) & p(0) \end{bmatrix}$$

let B be the standard basis for M_{22}, and let $B' = \{1, x, x^2\}$, $B'' = \{1, 1 + x, 1 + x^2\}$ be bases for P_2.

(a) Find $[T]_{B,B'}$ and $[T]_{B,B''}$.

(b) For the matrices obtained in part (a), compute $T(2 + 2x + x^2)$ using the three-step procedure illustrated in Example 2.

(c) Check the results obtained in part (b) by computing $T(2 + 2x + x^2)$ directly.

16. Let $T: M_{22} \to R^2$ be the linear transformation given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + b + c \\ d \end{bmatrix}$$

and let B be the standard basis for M_{22}, B' the standard basis for R^2, and

$$B'' = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\}$$

(a) Find $[T]_{B',B}$ and $[T]_{B'',B}$.

(b) Compute $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$ using the three-step procedure that was illustrated in Example 2 for both matrices found in part (a).

(c) Check the results obtained in part (b) by computing $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$ directly.

17. (***Calculus required***) Let $D: P_2 \to P_2$ be the differentiation operator $D(\mathbf{p}) = p'(x)$.

(a) Find the matrix for D relative to the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2 in which $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, $\mathbf{p}_3 = x^2$.

(b) Use the matrix in part (a) to compute $D(6 - 6x + 24x^2)$.

18. (***Calculus required***) Let $D: P_2 \to P_2$ be the differentiation operator $D(\mathbf{p}) = p'(x)$.

(a) Find the matrix for D relative to the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2 in which $\mathbf{p}_1 = 2$, $\mathbf{p}_2 = 2 - 3x$, $\mathbf{p}_3 = 2 - 3x + 8x^2$.

(b) Use the matrix in part (a) to compute $D(6 - 6x + 24x^2)$.

19. (***Calculus required***) Let V be the vector space of real-valued functions defined on the interval $(-\infty, \infty)$, and let $D: V \to V$ be the differentiation operator.

(a) Find the matrix for D relative to the basis $B = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ for V in which $\mathbf{f}_1 = 1$, $\mathbf{f}_2 = \sin x$, $\mathbf{f}_3 = \cos x$