6.3 Gram–Schmidt Process; *QR*-Decomposition

In many problems involving vector spaces, the problem solver is free to choose any basis for the vector space that seems appropriate. In inner product spaces, the solution of a problem can often be simplified by choosing a basis in which the vectors are orthogonal to one another. In this section we will show how such bases can be obtained.

Orthogonal and Orthonormal Sets

Recall from Section 6.2 that two vectors in an inner product space are said to be *orthogonal* if their inner product is zero. The following definition extends the notion of orthogonality to *sets* of vectors in an inner product space.

DEFINITION 1 A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

EXAMPLE 1 An Orthogonal Set in R^3

Let

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (1, 0, -1)$$

and assume that R^3 has the Euclidean inner product. It follows that the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal since $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$. ◀

It frequently happens that one has found a set of orthogonal vectors in an inner product space but what is actually needed is a set of *orthonormal* vectors. A simple way to convert an orthogonal set of nonzero vectors into an orthonormal set is to multiply each vector $\mathbf{v}$ in the orthogonal set by the reciprocal of its length to create a vector of norm 1 (called a ***unit vector***). To see why this works, suppose that $\mathbf{v}$ is a nonzero vector in an inner product space, and let

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \tag{1}$$

Note that Formula (1) is identical to Formula (4) of Section 3.2, but whereas Formula (4) was valid only for vectors in R^n with the Euclidean inner product, Formula (1) is valid in general inner product spaces.

Then it follows from Theorem 6.1.1(*b*) with $k = \|\mathbf{v}\|$ that

$$\|\mathbf{u}\| = \left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{1}{\cancel{\|\mathbf{v}\|}} \cancel{\|\mathbf{v}\|} = 1$$

This process of multiplying a vector $\mathbf{v}$ by the reciprocal of its length is called ***normalizing* $\mathbf{v}$**. We leave it as an exercise to show that normalizing the vectors in an orthogonal set of nonzero vectors preserves the orthogonality of the vectors and produces an orthonormal set.

EXAMPLE 2 Constructing an Orthonormal Set

The Euclidean norms of the vectors in Example 1 are

$$\|\mathbf{v}_1\| = 1, \quad \|\mathbf{v}_2\| = \sqrt{2}, \quad \|\mathbf{v}_3\| = \sqrt{2}$$

Consequently, normalizing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ yields

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (0, 1, 0), \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

EXAMPLE 1 An Orthogonal Set in R^3

Let

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (1, 0, -1)$$

and assume that R^3 has the Euclidean inner product. It follows that the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal since $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \langle \mathbf{v}_1, \mathbf{v}_3\rangle = \langle \mathbf{v}_2, \mathbf{v}_3\rangle = 0$. ◀

$S = \{v_1, v_2, v_3\}$ is orthogonal

$\langle v_1, v_2\rangle = (0)(1) + (1)(0) + (0)(1) = 0$

$\langle v_1, v_3\rangle = (0)(1) + (1)(0) + (0)(-1) = 0$

$\langle v_2, v_3\rangle = (1)(1) + (0)(0) + (1)(-1) = 1 + 0 - 1 = 0$

Example 2

$\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{0+1+0} = \sqrt{1} = 1$

$\|v_2\| = \sqrt{\langle v_2, v_2\rangle} = \sqrt{1+0+1} = \sqrt{2} \neq 1$

$\|v_3\| = \sqrt{\langle v_3, v_3\rangle} = \sqrt{1+0+1} = \sqrt{2} \neq 1$

$\therefore$ S is not orthonormal

$S = \{v_1, v_2, v_3\}$

$S' = \{u_1, u_2, u_3\}$ orthonormal set

$u_1 = v_1$

$u_2 = \frac{1}{\sqrt{2}} v_2 = \frac{1}{\sqrt{2}}(1, 0, 1) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$\|u_2\| = \sqrt{\langle u_2, u_2\rangle} = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = \sqrt{1} = 1$

$u_3 = \frac{1}{\sqrt{2}} v_3 = \frac{1}{\sqrt{2}}(1, 0, -1) = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

We leave it for you to verify that the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthonormal by showing that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0 \quad \text{and} \quad \|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1 \blacktriangleleft$$

In R^2 any two nonzero perpendicular vectors are linearly independent because neither is a scalar multiple of the other; and in R^3 any three nonzero mutually perpendicular vectors are linearly independent because no one lies in the plane of the other two (and hence is not expressible as a linear combination of the other two). The following theorem generalizes these observations.

THEOREM 6.3.1 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.*

Proof Assume that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0} \tag{2}$$

To demonstrate that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, we must prove that $k_1 = k_2 = \cdots = k_n = 0$.

For each $\mathbf{v}_i$ in S, it follows from (2) that

$$\langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle = \langle \mathbf{0}, \mathbf{v}_i \rangle = 0$$

or, equivalently,

$$k_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + k_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + k_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle = 0$$

From the orthogonality of S it follows that $\langle \mathbf{v}_j, \mathbf{v}_i \rangle = 0$ when $j \neq i$, so this equation reduces to

$$k_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 0$$

Since the vectors in S are assumed to be nonzero, it follows from the positivity axiom for inner products that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0$. Thus, the preceding equation implies that each k_i in Equation (2) is zero, which is what we wanted to prove. ◀

Since an orthonormal set is orthogonal, and since its vectors are nonzero (norm 1), it follows from Theorem 6.3.1 that every *orthonormal* set is linearly independent.

In an inner product space, a basis consisting of orthonormal vectors is called an ***orthonormal basis***, and a basis consisting of orthogonal vectors is called an ***orthogonal basis***. A familiar example of an orthonormal basis is the standard basis for R^n with the Euclidean inner product:

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

▶ EXAMPLE 3 An Orthonormal Basis for P_n

Recall from Example 7 of Section 6.1 that the standard inner product of the polynomials

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n \quad \text{and} \quad \mathbf{q} = b_0 + b_1x + \cdots + b_nx^n$$

is

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$$

$\langle 1, x^k \rangle = 0$, $k=1, \ldots n$

$\langle x^m, x^k \rangle = 0$, $k, m = 1, 2, \ldots n$

and the norm of $\mathbf{p}$ relative to this inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$$

You should be able to see from these formulas that the standard basis

$$S = \{1, x, x^2, \ldots, x^n\}$$

is orthonormal with respect to this inner product.

$\|1\| = 1$

$\|x\| = 1$

$\|x^2\| = 1$

$\vdots$

$\|x^n\| = 1$

example 2

$S = \{u_1, u_2, u_3\}$

EXAMPLE 4 An Orthonormal Basis

In Example 2 we showed that the vectors

$$\mathbf{u}_1 = (0, 1, 0), \quad \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \text{and} \quad \mathbf{u}_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

form an orthonormal set with respect to the Euclidean inner product on R^3. By Theorem 6.3.1, these vectors form a linearly independent set, and since R^3 is three-dimensional, it follows from Theorem 4.5.4 that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for R^3. ◀

Coordinates Relative to Orthonormal Bases

One way to express a vector $\mathbf{u}$ as a linear combination of basis vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

is to convert the vector equation

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

to a linear system and solve for the coefficients $c_1, c_2, \ldots, c_n$. However, if the basis happens to be orthogonal or orthonormal, then the following theorem shows that the coefficients can be obtained more simply by computing appropriate inner products.

THEOREM 6.3.2

(*a*) *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then*

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \tag{3}$$

(*b*) *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then*

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n\rangle\mathbf{v}_n \tag{4}$$

Proof (a) Since $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, every vector $\mathbf{u}$ in V can be expressed in the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

We will complete the proof by showing that

$$c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i\rangle}{\|\mathbf{v}_i\|^2} \tag{5}$$

for $i = 1, 2, \ldots, n$. To do this, observe first that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}_i\rangle &= \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \mathbf{v}_i\rangle \\ &= c_1\langle \mathbf{v}_1, \mathbf{v}_i\rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i\rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i\rangle\end{aligned}$$

Since S is an orthogonal set, all of the inner products in the last equality are zero except the ith, so we have

$$\langle \mathbf{u}, \mathbf{v}_i\rangle = c_i\langle \mathbf{v}_i, \mathbf{v}_i\rangle = c_i\|\mathbf{v}_i\|^2$$

Solving this equation for c_i yields (5), which completes the proof.

Proof (b) In this case, $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \cdots = \|\mathbf{v}_n\| = 1$, so Formula (3) simplifies to Formula (4). ◀

Using the terminology and notation from Definition 2 of Section 4.4, it follows from Theorem 6.3.2 that the coordinate vector of a vector $\mathbf{u}$ in V relative to an orthogonal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is

$$(\mathbf{u})_S = \left(\frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}, \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}, \dots, \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\right) \tag{6}$$

and relative to an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is

$$(\mathbf{u})_S = (\langle \mathbf{u}, \mathbf{v}_1\rangle, \langle \mathbf{u}, \mathbf{v}_2\rangle, \dots, \langle \mathbf{u}, \mathbf{v}_n\rangle) \tag{7}$$

EXAMPLE 5 A Coordinate Vector Relative to an Orthonormal Basis

Let

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = \left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right), \quad \mathbf{v}_3 = \left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right)$$

It is easy to check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for R^3 with the Euclidean inner product. Express the vector $\mathbf{u} = (1, 1, 1)$ as a linear combination of the vectors in S, and find the coordinate vector $(\mathbf{u})_S$.

Solution We leave it for you to verify that

$$\langle \mathbf{u}, \mathbf{v}_1\rangle = 1, \quad \langle \mathbf{u}, \mathbf{v}_2\rangle = -\tfrac{1}{5}, \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v}_3\rangle = \tfrac{7}{5}$$

Therefore, by Theorem 6.3.2 we have

$$\mathbf{u} = \mathbf{v}_1 - \tfrac{1}{5}\mathbf{v}_2 + \tfrac{7}{5}\mathbf{v}_3$$

that is,

$$(1, 1, 1) = (0, 1, 0) - \tfrac{1}{5}\left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right) + \tfrac{7}{5}\left(\tfrac{3}{5}, 0, \tfrac{4}{5}\right)$$

Thus, the coordinate vector of $\mathbf{u}$ relative to S is

$$(\mathbf{u})_S = (\langle \mathbf{u}, \mathbf{v}_1\rangle, \langle \mathbf{u}, \mathbf{v}_2\rangle, \langle \mathbf{u}, \mathbf{v}_3\rangle) = \left(1, -\tfrac{1}{5}, \tfrac{7}{5}\right)$$

EXAMPLE 6 An Orthonormal Basis from an Orthogonal Basis

(a) Show that the vectors

$$\mathbf{w}_1 = (0, 2, 0), \quad \mathbf{w}_2 = (3, 0, 3), \quad \mathbf{w}_3 = (-4, 0, 4)$$

form an orthogonal basis for R^3 with the Euclidean inner product, and use that basis to find an orthonormal basis by normalizing each vector.

(b) Express the vector $\mathbf{u} = (1, 2, 4)$ as a linear combination of the orthonormal basis vectors obtained in part (a).

Solution (a) The given vectors form an orthogonal set since

$$\langle \mathbf{w}_1, \mathbf{w}_2\rangle = 0, \quad \langle \mathbf{w}_1, \mathbf{w}_3\rangle = 0, \quad \langle \mathbf{w}_2, \mathbf{w}_3\rangle = 0$$

It follows from Theorem 6.3.1 that these vectors are linearly independent and hence form a basis for R^3 by Theorem 4.5.4. We leave it for you to calculate the norms of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ and then obtain the orthonormal basis

$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = (0, 1, 0), \quad \mathbf{v}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{v}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

Solution (b) It follows from Formula (4) that

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{u}, \mathbf{v}_3 \rangle \mathbf{v}_3$$

We leave it for you to confirm that

$$\langle \mathbf{u}, \mathbf{v}_1 \rangle = (1, 2, 4) \cdot (0, 1, 0) = 2$$
$$\langle \mathbf{u}, \mathbf{v}_2 \rangle = (1, 2, 4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = \frac{5}{\sqrt{2}}$$
$$\langle \mathbf{u}, \mathbf{v}_3 \rangle = (1, 2, 4) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = \frac{3}{\sqrt{2}}$$

and hence that

$$(1, 2, 4) = 2(0, 1, 0) + \frac{5}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) + \frac{3}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$ ◀

Orthogonal Projections

Many applied problems are best solved by working with orthogonal or orthonormal basis vectors. Such bases are typically found by starting with some simple basis (say a standard basis) and then converting that basis into an orthogonal or orthonormal basis. To explain exactly how that is done will require some preliminary ideas about orthogonal projections.

In Section 3.3 we proved a result called the *Projection Theorem* (see Theorem 3.3.2) that dealt with the problem of decomposing a vector $\mathbf{u}$ in R^n into a sum of two terms, $\mathbf{w}_1$ and $\mathbf{w}_2$, in which $\mathbf{w}_1$ is the orthogonal projection of $\mathbf{u}$ on some nonzero vector $\mathbf{a}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{w}_1$ (Figure 3.3.2). That result is a special case of the following more general theorem, which we will state without proof.

THEOREM 6.3.3 Projection Theorem

If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 \tag{8}$$

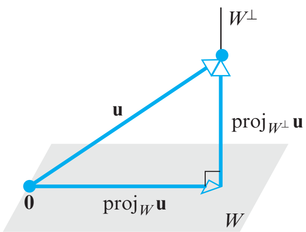
where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$.

The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in Formula (8) are commonly denoted by

$$\mathbf{w}_1 = \operatorname{proj}_W \mathbf{u} \quad \text{and} \quad \mathbf{w}_2 = \operatorname{proj}_{W^{\perp}} \mathbf{u} \tag{9}$$

These are called the ***orthogonal projection of* u *on* W** and the ***orthogonal projection of* u *on* $W^{\perp}$**, respectively. The vector $\mathbf{w}_2$ is also called the ***component of* u *orthogonal to* W**. Using the notation in (9), Formula (8) can be expressed as

$$\mathbf{u} = \operatorname{proj}_W \mathbf{u} + \operatorname{proj}_{W^{\perp}} \mathbf{u} \tag{10}$$



▲ Figure 6.3.1

(Figure 6.3.1). Moreover, since $\operatorname{proj}_{W^{\perp}} \mathbf{u} = \mathbf{u} - \operatorname{proj}_W \mathbf{u}$, we can also express Formula (10) as

$$\mathbf{u} = \operatorname{proj}_W \mathbf{u} + (\mathbf{u} - \operatorname{proj}_W \mathbf{u}) \tag{11}$$

The following theorem provides formulas for calculating orthogonal projections.

Although Formulas (12) and (13) are expressed in terms of orthogonal and orthonormal basis vectors, the resulting vector $\text{proj}_W \mathbf{u}$ does not depend on the basis vectors that are used.

THEOREM 6.3.4 *Let W be a finite-dimensional subspace of an inner product space V.*

(*a*) *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then*

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r \tag{12}$$

(*b*) *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W, and $\mathbf{u}$ is any vector in V, then*

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r \tag{13}$$

***Proof* (*a*)** It follows from Theorem 6.3.3 that the vector $\mathbf{u}$ can be expressed in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 = \text{proj}_W \mathbf{u}$ is in W and $\mathbf{w}_2$ is in $W^\perp$; and it follows from Theorem 6.3.2 that the component $\text{proj}_W \mathbf{u} = \mathbf{w}_1$ can be expressed in terms of the basis vectors for W as

$$\text{proj}_W \mathbf{u} = \mathbf{w}_1 = \frac{\langle \mathbf{w}_1, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{w}_1, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{w}_1, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r \tag{14}$$

Since $\mathbf{w}_2$ is orthogonal to W, it follows that

$$\langle \mathbf{w}_2, \mathbf{v}_1 \rangle = \langle \mathbf{w}_2, \mathbf{v}_2 \rangle = \cdots = \langle \mathbf{w}_2, \mathbf{v}_r \rangle = 0$$

so we can rewrite (14) as

$$\text{proj}_W \mathbf{u} = \mathbf{w}_1 = \frac{\langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r$$

or, equivalently, as

$$\text{proj}_W \mathbf{u} = \mathbf{w}_1 = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r$$

***Proof* (*b*)** In this case, $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \cdots = \|\mathbf{v}_r\| = 1$, so Formula (14) simplifies to Formula (13). ◀

▶ EXAMPLE 7 Calculating Projections

Let R^3 have the Euclidean inner product, and let W be the subspace spanned by the orthonormal vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$. From Formula (13) the orthogonal projection of $\mathbf{u} = (1, 1, 1)$ on W is

$$\begin{aligned} \text{proj}_W \mathbf{u} &= \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 \\ &= (1)(0, 1, 0) + \left(-\tfrac{1}{5}\right)\left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right) \\ &= \left(\tfrac{4}{25}, 1, -\tfrac{3}{25}\right) \end{aligned}$$

The component of $\mathbf{u}$ orthogonal to W is

$$\text{proj}_{W^\perp} \mathbf{u} = \mathbf{u} - \text{proj}_W \mathbf{u} = (1, 1, 1) - \left(\tfrac{4}{25}, 1, -\tfrac{3}{25}\right) = \left(\tfrac{21}{25}, 0, \tfrac{28}{25}\right)$$

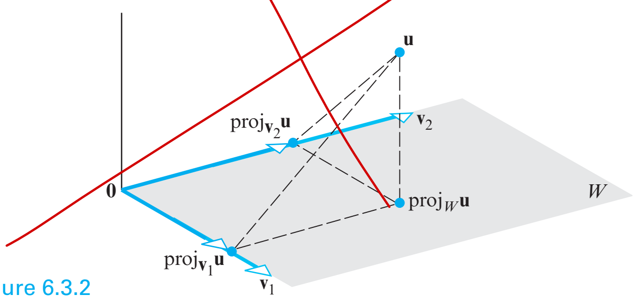
Observe that $\text{proj}_{W^\perp} \mathbf{u}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, so this vector is orthogonal to each vector in the space W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, as it should be. ◀

A Geometric Interpretation of Orthogonal Projections

If W is a one-dimensional subspace of an inner product space V, say $\text{span}\{\mathbf{a}\}$, then Formula (12) has only the one term

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{a} \rangle}{\|\mathbf{a}\|^2}\mathbf{a}$$

In the special case where V is R^3 with the Euclidean inner product, this is exactly Formula (10) of Section 3.3 for the orthogonal projection of $\mathbf{u}$ along $\mathbf{a}$. This suggests that we can think of (12) as the sum of orthogonal projections on "axes" determined by the basis vectors for the subspace W (Figure 6.3.2).



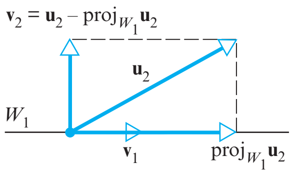
▶ Figure 6.3.2

The Gram–Schmidt Process

We have seen that orthonormal bases exhibit a variety of useful properties. Our next theorem, which is the main result in this section, shows that every nonzero finite-dimensional vector space has an orthonormal basis. The proof of this result is extremely important since it provides an algorithm, or method, for converting an arbitrary basis into an orthonormal basis.

THEOREM 6.3.5 *Every nonzero finite-dimensional inner product space has an orthonormal basis.*

Proof Let W be any nonzero finite-dimensional subspace of an inner product space, and suppose that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ is any basis for W. It suffices to show that W has an orthogonal basis since the vectors in that basis can be normalized to obtain an orthonormal basis. The following sequence of steps will produce an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ for W:



▲ Figure 6.3.3

Step 1. Let $\mathbf{v}_1 = \mathbf{u}_1$.

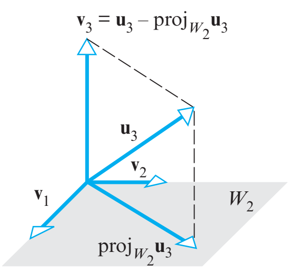
Step 2. As illustrated in Figure 6.3.3, we can obtain a vector $\mathbf{v}_2$ that is orthogonal to $\mathbf{v}_1$ by computing the component of $\mathbf{u}_2$ that is orthogonal to the space W_1 spanned by $\mathbf{v}_1$. Using Formula (12) to perform this computation, we obtain

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

Of course, if $\mathbf{v}_2 = \mathbf{0}$, then $\mathbf{v}_2$ is not a basis vector. But this cannot happen, since it would then follow from the preceding formula for $\mathbf{v}_2$ that

$$\mathbf{u}_2 = \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$$

which implies that $\mathbf{u}_2$ is a multiple of $\mathbf{u}_1$, contradicting the linear independence of the basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$.



▲ Figure 6.3.4

Step 3. To construct a vector $\mathbf{v}_3$ that is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, we compute the component of $\mathbf{u}_3$ orthogonal to the space W_2 spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ (Figure 6.3.4). Using Formula (12) to perform this computation, we obtain

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

As in Step 2, the linear independence of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ ensures that $\mathbf{v}_3 \neq \mathbf{0}$. We leave the details for you.

Step 4. To determine a vector $\mathbf{v}_4$ that is orthogonal to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we compute the component of $\mathbf{u}_4$ orthogonal to the space W_3 spanned by $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. From (12),

$$\mathbf{v}_4 = \mathbf{u}_4 - \operatorname{proj}_{W_3} \mathbf{u}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$$

Continuing in this way we will produce after r steps an orthogonal set of nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$. Since such sets are linearly independent, we will have produced an orthogonal basis for the r-dimensional space W. By normalizing these basis vectors we can obtain an orthonormal basis. ◀

The step-by-step construction of an orthogonal (or orthonormal) basis given in the foregoing proof is called the ***Gram–Schmidt process***. For reference, we provide the following summary of the steps.

The Gram–Schmidt Process

To convert a basis $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$, perform the following computations:

Step 1. $\mathbf{v}_1 = \mathbf{u}_1$

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

Step 4. $\mathbf{v}_4 = \mathbf{u}_4 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

$\vdots$

(continue for r steps)

Optional Step. To convert the orthogonal basis into an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_r\}$, normalize the orthogonal basis vectors.

Jorgen Pederson Gram (1850–1916)

Historical Note Erhardt Schmidt (1875–1959) was a German mathematician who studied for his doctoral degree at Göttingen University under David Hilbert, one of the giants of modern mathematics. For most of his life he taught at Berlin University where, in addition to making important contributions to many branches of mathematics, he fashioned some of Hilbert's ideas into a general concept, called a *Hilbert space*—a fundamental structure in the study of infinite-dimensional vector spaces. He first described the process that bears his name in a paper on integral equations that he published in 1907.

Historical Note Gram was a Danish actuary whose early education was at village schools supplemented by private tutoring. He obtained a doctorate degree in mathematics while working for the Hafnia Life Insurance Company, where he specialized in the mathematics of accident insurance. It was in his dissertation that his contributions to the Gram–Schmidt process were formulated. He eventually became interested in abstract mathematics and received a gold medal from the Royal Danish Society of Sciences and Letters in recognition of his work. His lifelong interest in applied mathematics never wavered, however, and he produced a variety of treatises on Danish forest management.

[*Image: http://www-history.mcs.st-and.ac.uk/PictDisplay/Gram.html*]

▶ EXAMPLE 8 Using the Gram–Schmidt Process

Assume that the vector space R^3 has the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = (1, 1, 1), \quad \mathbf{u}_2 = (0, 1, 1), \quad \mathbf{u}_3 = (0, 0, 1)$$

into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

Solution

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1} \mathbf{u}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

$$= (0, 1, 1) - \frac{2}{3}(1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2} \mathbf{u}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

$$= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Thus,

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

form an orthogonal basis for R^3. The norms of these vectors are

$$\|\mathbf{v}_1\| = \sqrt{3}, \quad \|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}, \quad \|\mathbf{v}_3\| = \frac{1}{\sqrt{2}}$$

so an orthonormal basis for R^3 is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \blacktriangleleft$$

Remark In the last example we normalized at the end to convert the orthogonal basis into an orthonormal basis. Alternatively, we could have normalized each orthogonal basis vector as soon as it was obtained, thereby producing an orthonormal basis step by step. However, that procedure generally has the disadvantage in hand calculation of producing more square roots to manipulate. A more useful variation is to "scale" the orthogonal basis vectors at each step to eliminate some of the fractions. For example, after Step 2 above, we could have multiplied by 3 to produce $(-2, 1, 1)$ as the second orthogonal basis vector, thereby simplifying the calculations in Step 3.

CALCULUS REQUIRED

▶ EXAMPLE 9 Legendre Polynomials

Let the vector space P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\, dx$$

Apply the Gram–Schmidt process to transform the standard basis $\{1, x, x^2\}$ for P_2 into an orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$.

EXAMPLE 8 Using the Gram–Schmidt Process

Assume that the vector space R^3 has the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = (1, 1, 1), \quad \mathbf{u}_2 = (0, 1, 1), \quad \mathbf{u}_3 = (0, 0, 1)$$

into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

step 1: $\boxed{v_1 = u_1 = (1,1,1)}$

step 2: $v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$

$\langle u_2, v_1 \rangle = \langle (0,1,1), (1,1,1) \rangle = (0)(1) + (1)(1) + (1)(1) = 2$

$\|v_1\|^2 = \langle v_1, v_1 \rangle = \langle (1,1,1), (1,1,1) \rangle = 3$

$v_2 = (0,1,1) - \dfrac{2}{3}(1,1,1) = (0,1,1) - \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$

$\boxed{v_2 = \left(\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)}$ *

step 3: $v_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle}{\boxed{\|v_1\|^2}} v_1 - \dfrac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

$\langle u_3, v_1 \rangle = \langle (0,0,1), (1,1,1) \rangle = \boxed{1}$

$\langle u_3, v_2 \rangle = \langle (0,0,1), \left(\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right) \rangle = \boxed{\dfrac{1}{3}}$

$\|v_2\|^2 = \langle \left(\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right), \left(\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right) \rangle = \dfrac{4}{9} + \dfrac{1}{9} + \dfrac{1}{9} = \dfrac{6}{9} = \boxed{\dfrac{2}{3}}$

$v_3 = (0,0,1) - \dfrac{1}{3}(1,1,1) - \dfrac{1}{\cancel{3}} \cdot \dfrac{\cancel{3}}{2}\left(\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

$= (0,0,1) - \left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right) - \left(\dfrac{-1}{3}, \dfrac{1}{6}, \dfrac{1}{6}\right)$

$= \left(-\dfrac{1}{3}, -\dfrac{1}{3}, \dfrac{2}{3}\right) - \left(-\dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{6}\right) = \boxed{\left(0, -\dfrac{1}{2}, \dfrac{1}{2}\right)}$ v_3

$v_1 = (1,1,1)$, $v_2 = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $v_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

$S' = \{v_1, v_2, v_3\}$ orthogonal set

$\|v_1\| = \sqrt{1+1+1} = \sqrt{3}$

$\|v_2\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$

$\|v_3\| = \sqrt{0 + \frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$

$\frac{1}{\|v_2\|} v_2 = \frac{3}{\sqrt{6}}\left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$S'' = \left\{ q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right.$

$\left. q_3 = \frac{v_3}{\|v_3\|} = v_3 = \left(0, \frac{-1}{2}, \frac{1}{2}\right) \right\}$

S'' is orthonormal set

EXAMPLE 9 Legendre Polynomials

Let the vector space P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Apply the Gram–Schmidt process to transform the standard basis $\{1, x, x^2\}$ for P_2 into an orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$.

$u_1 \; u_2 \; u_3$

step 1: $\phi_1 = u_1 = 1$

step 2: $\phi_2 = u_2 - \frac{\langle u_2, \phi_1 \rangle}{\|\phi_1\|^2} \cdot \phi_1$

$\langle u_2, \phi_1 \rangle = \int_{-1}^{1} x \cdot 1\,dx = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

$\phi_2 = u_2 = x$

step 3: $\phi_3 = u_3 - \frac{\langle u_3, \phi_1 \rangle}{\|\phi_1\|^2} \cdot \phi_1 - \frac{\langle u_3, \phi_2 \rangle}{\|\phi_2\|^2} \cdot \phi_2$

$$\langle u_3, \varphi_1 \rangle = \int_{-1}^{1} x^2 \cdot 1 \, dx = \int_{-1}^{1} x^2 dx = \frac{x^3}{3} \Big|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \boxed{\frac{2}{3}}$$

$$\|\varphi_1\|^2 = \langle \varphi_1, \varphi_1 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = x \Big|_{-1}^{1} = 1 + 1 = 2$$

$$\langle u_3, \varphi_2 \rangle = \int_{-1}^{1} x^2 \cdot x \, dx = \int_{-1}^{1} x^3 dx = \frac{x^4}{4} \Big|_{-1}^{1} = 0$$

$$\therefore \varphi_3 = u_3 - \frac{\langle u_3, \varphi_1 \rangle}{\|\varphi_1\|^2} \varphi_1 - 0$$

$$= x^2 - \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} (1) = \boxed{x^2 - \frac{1}{3}}$$

$$\{\varphi_1, \varphi_2, \varphi_3\} = \left\{1, x, x^2 - \frac{1}{3}\right\}$$

Solution Take $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x$, and $\mathbf{u}_3 = x^2$.

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = 1$

Step 2. We have

$$\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = \int_{-1}^{1} x\, dx = 0$$

so

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \mathbf{u}_2 = x$$

Step 3. We have

$$\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = \int_{-1}^{1} x^2\, dx = \frac{x^3}{3}\bigg]_{-1}^{1} = \frac{2}{3}$$

$$\langle \mathbf{u}_3, \mathbf{v}_2 \rangle = \int_{-1}^{1} x^3\, dx = \frac{x^4}{4}\bigg]_{-1}^{1} = 0$$

$$\|\mathbf{v}_1\|^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^{1} 1\, dx = x\bigg]_{-1}^{1} = 2$$

so

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = x^2 - \frac{1}{3}$$

Thus, we have obtained the orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$ in which

$$\phi_1(x) = 1, \quad \phi_2(x) = x, \quad \phi_3(x) = x^2 - \frac{1}{3}$$ ◀

Remark The orthogonal basis vectors in the last example are often scaled so all three functions have a value of 1 at $x = 1$. The resulting polynomials

$$1, \quad x, \quad \frac{1}{2}(3x^2 - 1)$$

which are known as the first three ***Legendre polynomials***, play an important role in a variety of applications. The scaling does not affect the orthogonality.

Extending Orthonormal Sets to Orthonormal Bases

Recall from part (*b*) of Theorem 4.5.5 that a linearly independent set in a finite-dimensional vector space can be enlarged to a basis by adding appropriate vectors. The following theorem is an analog of that result for orthogonal and orthonormal sets in finite-dimensional inner product spaces.

THEOREM 6.3.6 *If W is a finite-dimensional inner product space, then:*

(*a*) *Every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W.*

(*b*) *Every orthonormal set in W can be enlarged to an orthonormal basis for W.*

We will prove part (*b*) and leave part (*a*) as an exercise.

Proof (*b*) Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ is an orthonormal set of vectors in W. Part (*b*) of Theorem 4.5.5 tells us that we can enlarge S to some basis

$$S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s, \mathbf{v}_{s+1}, \ldots, \mathbf{v}_k\}$$

for W. If we now apply the Gram–Schmidt process to the set S', then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$, will not be affected since they are already orthonormal, and the resulting set

$$S'' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s, \mathbf{v}'_{s+1}, \ldots, \mathbf{v}'_k\}$$

will be an orthonormal basis for W. ◀

OPTIONAL

QR-Decomposition

In recent years a numerical algorithm based on the Gram–Schmidt process, and known as ***QR-decomposition***, has assumed growing importance as the mathematical foundation for a wide variety of numerical algorithms, including those for computing eigenvalues of large matrices. The technical aspects of such algorithms are discussed in textbooks that specialize in the numerical aspects of linear algebra. However, we will discuss some of the underlying ideas here. We begin by posing the following problem.

> **Problem** If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram–Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ and that Q has orthonormal column vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$. Thus, A and Q can be written in partitioned form as

$$A = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n] \quad \text{and} \quad Q = [\mathbf{q}_1 \mid \mathbf{q}_2 \mid \cdots \mid \mathbf{q}_n]$$

It follows from Theorem 6.3.2(b) that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are expressible in terms of the vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ as

$$\begin{aligned}
\mathbf{u}_1 &= \langle \mathbf{u}_1, \mathbf{q}_1\rangle \mathbf{q}_1 + \langle \mathbf{u}_1, \mathbf{q}_2\rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_1, \mathbf{q}_n\rangle \mathbf{q}_n \\
\mathbf{u}_2 &= \langle \mathbf{u}_2, \mathbf{q}_1\rangle \mathbf{q}_1 + \langle \mathbf{u}_2, \mathbf{q}_2\rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_2, \mathbf{q}_n\rangle \mathbf{q}_n \\
&\;\;\vdots \\
\mathbf{u}_n &= \langle \mathbf{u}_n, \mathbf{q}_1\rangle \mathbf{q}_1 + \langle \mathbf{u}_n, \mathbf{q}_2\rangle \mathbf{q}_2 + \cdots + \langle \mathbf{u}_n, \mathbf{q}_n\rangle \mathbf{q}_n
\end{aligned}$$

Recalling from Section 1.3 (Example 9) that the jth column vector of a matrix product is a linear combination of the column vectors of the first factor with coefficients coming from the jth column of the second factor, it follows that these relationships can be expressed in matrix form as

$$[\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n] = [\mathbf{q}_1 \mid \mathbf{q}_2 \mid \cdots \mid \mathbf{q}_n] \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1\rangle \\ \langle \mathbf{u}_1, \mathbf{q}_2\rangle & \langle \mathbf{u}_2, \mathbf{q}_2\rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2\rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{u}_1, \mathbf{q}_n\rangle & \langle \mathbf{u}_2, \mathbf{q}_n\rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n\rangle \end{bmatrix}$$

or more briefly as

$$A = QR \tag{15}$$

where R is the second factor in the product. However, it is a property of the Gram–Schmidt process that for $j \geq 2$, the vector $\mathbf{q}_j$ is orthogonal to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{j-1}$. Thus, all entries below the main diagonal of R are zero, and R has the form

$$R = \begin{array}{cl} \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_1\rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle & \cdots & \langle \mathbf{u}_n, \mathbf{q}_2\rangle \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{u}_n, \mathbf{q}_n\rangle \end{bmatrix} & \begin{array}{l} \rightarrow q_1 \\ \rightarrow q_2 \\ \vdots \\ \rightarrow q_n \end{array} \end{array} \tag{16}$$

We leave it for you to show that R is invertible by showing that its diagonal entries are nonzero. Thus, Equation (15) is a factorization of A into the product of a matrix Q

with orthonormal column vectors and an invertible upper triangular matrix R. We call Equation (15) a ***QR-decomposition of A***. In summary, we have the following theorem.

It is common in numerical linear algebra to say that a matrix with linearly independent columns has ***full column rank***.

THEOREM 6.3.7 ***QR*-Decomposition**

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Recall from Theorem 5.1.5 (the Equivalence Theorem) that a *square* matrix has linearly independent column vectors if and only if it is invertible. Thus, it follows from Theorem 6.3.7 that *every invertible matrix has a QR-decomposition.*

EXAMPLE 10 ***QR*-Decomposition of a 3 × 3 Matrix**

Find a QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution The column vectors of A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Applying the Gram–Schmidt process with normalization to these column vectors yields the orthonormal vectors (see Example 8)

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, it follows from Formula (16) that R is

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle & \langle \mathbf{u}_3, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \langle \mathbf{u}_3, \mathbf{q}_2 \rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3 \rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

from which it follows that a QR-decomposition of A is

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underset{=}{=} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R}$$

▶ **EXAMPLE 10** ***QR*-Decomposition of a 3 × 3 Matrix**

Find a QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Gram-smith process on u_1, u_2, u_3

$$S'' = \left\{ q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \; q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \right.$$

$$\left. q_3 = \frac{v_3}{\|v_3\|} = v_3 = \left(0, \frac{-1}{2}, \frac{1}{2}\right) \right\}$$

$$\therefore Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{2} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}$$

$$Q \cdot R = A$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$$

$\langle u_1, q_1 \rangle = \frac{3}{\sqrt{3}}$, $\langle u_2, q_2 \rangle = \frac{2}{\sqrt{6}}$

$\langle u_2, q_1 \rangle = \frac{2}{\sqrt{3}}$ $\quad \langle u_3, q_2 \rangle = \frac{1}{\sqrt{6}}$

$\langle u_3, q_1 \rangle = \frac{1}{\sqrt{3}}$ $\quad \langle u_3, q_3 \rangle = \frac{1}{2}$

$$\therefore R = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Exercise Set 6.3

1. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^2.

(a) $(0, 1),\ (2, 0)$

(b) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(d) $(0, 0),\ (0, 1)$

2. In each part, determine whether the set of vectors is orthogonal and whether it is orthonormal with respect to the Euclidean inner product on R^3.

(a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

(b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right),\ \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right),\ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

(c) $(1, 0, 0),\ \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ (0, 0, 1)$

(d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right),\ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

3. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on P_2 (see Example 7 of Section 6.1).

(a) $p_1(x) = \frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2,\ \ p_2(x) = \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2,$
$p_3(x) = \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2$

(b) $p_1(x) = 1,\ \ p_2(x) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2,\ \ p_3(x) = x^2$

4. In each part, determine whether the set of vectors is orthogonal with respect to the standard inner product on M_{22} (see Example 6 of Section 6.1).

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix},\ \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix},\ \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

In Exercises 5–6, show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space.

5. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} \frac{1}{5} & -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & 0 & -\frac{2}{3} \end{bmatrix}$

7. Verify that the vectors

$$\mathbf{v}_1 = \left(-\tfrac{3}{5}, \tfrac{4}{5}, 0\right),\ \ \mathbf{v}_2 = \left(\tfrac{4}{5}, \tfrac{3}{5}, 0\right),\ \ \mathbf{v}_3 = (0, 0, 1)$$

form an orthonormal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (1, -2, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

8. Use Theorem 6.3.2(*b*) to express the vector $\mathbf{u} = (3, -7, 4)$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in Exercise 7.

9. Verify that the vectors

$$\mathbf{v}_1 = (2, -2, 1),\quad \mathbf{v}_2 = (2, 1, -2),\quad \mathbf{v}_3 = (1, 2, 2)$$

form an orthogonal basis for R^3 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*a*) to express the vector $\mathbf{u} = (-1, 0, 2)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

10. Verify that the vectors

$$\mathbf{v}_1 = (1, -1, 2, -1),\quad \mathbf{v}_2 = (-2, 2, 3, 2),$$
$$\mathbf{v}_3 = (1, 2, 0, -1),\quad \mathbf{v}_4 = (1, 0, 0, 1)$$

form an orthogonal basis for R^4 with respect to the Euclidean inner product, and then use Theorem 6.3.2(*a*) to express the vector $\mathbf{u} = (1, 1, 1, 1)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

In Exercises 11–14, find the coordinate vector $(\mathbf{u})_S$ for the vector $\mathbf{u}$ and the basis S that were given in the stated exercise.

11. Exercise 7

12. Exercise 8

13. Exercise 9

14. Exercise 10

In Exercises 15–18, let R^2 have the Euclidean inner product.

(a) Find the orthogonal projection of $\mathbf{u}$ onto the line spanned by the vector $\mathbf{v}$.

(b) Find the component of $\mathbf{u}$ orthogonal to the line spanned by the vector $\mathbf{v}$, and confirm that this component is orthogonal to the line.

15. $\mathbf{u} = (-1, 6);\ \ \mathbf{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$

16. $\mathbf{u} = (2, 3);\ \ \mathbf{v} = \left(\frac{5}{13}, \frac{12}{13}\right)$

17. $\mathbf{u} = (2, 3);\ \ \mathbf{v} = (1, 1)$

18. $\mathbf{u} = (3, -1);\ \ \mathbf{v} = (3, 4)$

In Exercises 19–22, let R^3 have the Euclidean inner product.

(a) Find the orthogonal projection of $\mathbf{u}$ onto the plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Find the component of $\mathbf{u}$ orthogonal to the plane spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and confirm that this component is orthogonal to the plane.

19. $\mathbf{u} = (4, 2, 1);\ \ \mathbf{v}_1 = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right),\ \ \mathbf{v}_2 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

20. $\mathbf{u} = (3, -1, 2);\ \ \mathbf{v}_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right),\ \ \mathbf{v}_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

21. $\mathbf{u} = (1, 0, 3);\ \ \mathbf{v}_1 = (1, -2, 1),\ \ \mathbf{v}_2 = (2, 1, 0)$

22. $\mathbf{u} = (1, 0, 2);\ \ \mathbf{v}_1 = (3, 1, 2),\ \ \mathbf{v}_2 = (-1, 1, 1)$

In Exercises 23–24, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1, 2, 0, -2)$ on the subspace W spanned by these vectors.

23. $\mathbf{v}_1 = (1, 1, 1, 1),\ \ \mathbf{v}_2 = (1, 1, -1, -1)$

24. $\mathbf{v}_1 = (0, 1, -4, -1),\ \ \mathbf{v}_2 = (3, 5, 1, 1)$

In Exercises 25–26, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are orthonormal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1, 2, 0, -1)$ onto the subspace W spanned by these vectors.

25. $\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$,
$\mathbf{v}_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)$

26. $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{v}_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$,
$\mathbf{v}_3 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

In Exercises 27–28, let R^2 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ into an orthonormal basis. Draw both sets of basis vectors in the xy-plane.

27. $\mathbf{u}_1 = (1, -3)$, $\mathbf{u}_2 = (2, 2)$ **28.** $\mathbf{u}_1 = (1, 0)$, $\mathbf{u}_2 = (3, -5)$

In Exercises 29–30, let R^3 have the Euclidean inner product and use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis.

29. $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (-1, 1, 0)$, $\mathbf{u}_3 = (1, 2, 1)$

30. $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (3, 7, -2)$, $\mathbf{u}_3 = (0, 4, 1)$

31. Let R^4 have the Euclidean inner product. Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.

$$\mathbf{u}_1 = (0, 2, 1, 0), \quad \mathbf{u}_2 = (1, -1, 0, 0),$$
$$\mathbf{u}_3 = (1, 2, 0, -1), \quad \mathbf{u}_4 = (1, 0, 0, 1)$$

32. Let R^3 have the Euclidean inner product. Find an orthonormal basis for the subspace spanned by $(0, 1, 2)$, $(-1, 0, 1)$, $(-1, 1, 3)$.

33. Let $\mathbf{b}$ and W be as in Exercise 23. Find vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

34. Let $\mathbf{b}$ and W be as in Exercise 25. Find vectors $\mathbf{w}_1$ in W and $\mathbf{w}_2$ in $W^\perp$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

35. Let R^3 have the Euclidean inner product. The subspace of R^3 spanned by the vectors $\mathbf{u}_1 = (1, 1, 1)$ and $\mathbf{u}_2 = (2, 0, -1)$ is a plane passing through the origin. Express $\mathbf{w} = (1, 2, 3)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ lies in the plane and $\mathbf{w}_2$ is perpendicular to the plane.

36. Let R^4 have the Euclidean inner product. Express the vector $\mathbf{w} = (-1, 2, 6, 0)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in the space W spanned by $\mathbf{u}_1 = (-1, 0, 1, 2)$ and $\mathbf{u}_2 = (0, 1, 0, 1)$, and $\mathbf{w}_2$ is orthogonal to W.

37. Let R^3 have the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$$

Use the Gram–Schmidt process to transform $\mathbf{u}_1 = (1, 1, 1)$, $\mathbf{u}_2 = (1, 1, 0)$, $\mathbf{u}_3 = (1, 0, 0)$ into an orthonormal basis.

38. Verify that the set of vectors $\{(1, 0), (0, 1)\}$ is orthogonal with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + u_2v_2$ on R^2; then convert it to an orthonormal set by normalizing the vectors.

39. Find vectors $\mathbf{x}$ and $\mathbf{y}$ in R^2 that are orthonormal with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ but are not orthonormal with respect to the Euclidean inner product.

40. In Example 3 of Section 4.9 we found the orthogonal projection of the vector $\mathbf{x} = (1, 5)$ onto the line through the origin making an angle of $\pi/6$ radians with the positive x-axis. Solve that same problem using Theorem 6.3.4.

41. This exercise illustrates that the orthogonal projection resulting from Formula (12) in Theorem 6.3.4 does not depend on which orthogonal basis vectors are used.

(a) Let R^3 have the Euclidean inner product, and let W be the subspace of R^3 spanned by the orthogonal vectors

$$\mathbf{v}_1 = (1, 0, 1) \quad \text{and} \quad \mathbf{v}_2 = (0, 1, 0)$$

Show that the orthogonal vectors

$$\mathbf{v}_1' = (1, 1, 1) \quad \text{and} \quad \mathbf{v}_2' = (1, -2, 1)$$

span the same subspace W.

(b) Let $\mathbf{u} = (-3, 1, 7)$ and show that the same vector $\text{proj}_W \mathbf{u}$ results regardless of which of the bases in part (a) is used for its computation.

42. **(*Calculus required*)** Use Theorem 6.3.2(*a*) to express the following polynomials as linear combinations of the first three Legendre polynomials (see the Remark following Example 9).

(a) $1 + x + 4x^2$ (b) $2 - 7x^2$ (c) $4 + 3x$

43. **(*Calculus required*)** Let P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\,dx$$

Apply the Gram–Schmidt process to transform the standard basis $S = \{1, x, x^2\}$ into an orthonormal basis.

44. Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 6 & 1 & -5 \\ 2 & 1 & 1 \\ -2 & -2 & 5 \\ 6 & 8 & -7 \end{bmatrix}$$

In Exercises 45–48, we obtained the column vectors of Q by applying the Gram–Schmidt process to the column vectors of A. Find a QR-decomposition of the matrix A.

45. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

46. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

47. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix}$

49. Find a QR-decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

50. In the Remark following Example 8 we discussed two alternative ways to perform the calculations in the Gram–Schmidt process: normalizing each orthogonal basis vector as soon as it is calculated and scaling the orthogonal basis vectors at each step to eliminate fractions. Try these methods in Example 8.

Working with Proofs

51. Prove part (a) of Theorem 6.3.6.

52. In Step 3 of the proof of Theorem 6.3.5, it was stated that "the linear independence of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ ensures that $\mathbf{v}_3 \neq \mathbf{0}$." Prove this statement.

53. Prove that the diagonal entries of R in Formula (16) are nonzero.

54. Show that matrix Q in Example 10 has the property $QQ^T = I_3$, and prove that every $m \times n$ matrix Q with orthonormal column vectors has the property $QQ^T = I_m$.

55. (a) Prove that if W is a subspace of a finite-dimensional vector space V, then the mapping $T: V \to W$ defined by $T(\mathbf{v}) = \text{proj}_W \mathbf{v}$ is a linear transformation.

(b) What are the range and kernel of the transformation in part (a)?

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) Every linearly independent set of vectors in an inner product space is orthogonal.

(b) Every orthogonal set of vectors in an inner product space is linearly independent.

(c) Every nontrivial subspace of R^3 has an orthonormal basis with respect to the Euclidean inner product.

(d) Every nonzero finite-dimensional inner product space has an orthonormal basis.

(e) $\text{proj}_W \mathbf{x}$ is orthogonal to every vector of W.

(f) If A is an $n \times n$ matrix with a nonzero determinant, then A has a QR-decomposition.

Working with Technology

T1. (a) Use the Gram–Schmidt process to find an orthonormal basis relative to the Euclidean inner product for the column space of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & -1 & 1 & 1 \end{bmatrix}$$

(b) Use the method of Example 9 to find a QR-decomposition of A.

T2. Let P_4 have the evaluation inner product at the points $-2, -1, 0, 1, 2$. Find an orthogonal basis for P_4 relative to this inner product by applying the Gram–Schmidt process to the vectors

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \quad \mathbf{p}_3 = x^3, \quad \mathbf{p}_4 = x^4$$

6.4 Best Approximation; Least Squares

There are many applications in which some linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns should be consistent on physical grounds but fails to be so because of measurement errors in the entries of A or $\mathbf{b}$. In such cases one looks for vectors that come as close as possible to being solutions in the sense that they minimize $\|\mathbf{b} - A\mathbf{x}\|$ with respect to the Euclidean inner product on R^m. In this section we will discuss methods for finding such minimizing vectors.

Least Squares Solutions of Linear Systems

Suppose that $A\mathbf{x} = \mathbf{b}$ is an *inconsistent* linear system of m equations in n unknowns in which we suspect the inconsistency to be caused by errors in the entries of A or $\mathbf{b}$. Since no exact solution is possible, we will look for a vector $\mathbf{x}$ that comes as "close as possible" to being a solution in the sense that it minimizes $\|\mathbf{b} - A\mathbf{x}\|$ with respect to the Euclidean