Working with Technology

T1. (a) Confirm that the following matrix generates an inner product.

$$A = \begin{bmatrix} 5 & 8 & 6 & -13 \\ 3 & -1 & 0 & -9 \\ 0 & 1 & -1 & 0 \\ 2 & 4 & 3 & -5 \end{bmatrix}$$

(b) For the following vectors, use the inner product in part (a) to compute $\langle \mathbf{u}, \mathbf{v} \rangle$, first by Formula (5) and then by Formula (6).

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

T2. Let the vector space P_4 have the evaluation inner product at the points

$$-2, \quad -1, \quad 0, \quad 1, \quad 2$$

and let

$$\mathbf{p} = p(x) = x + x^3 \quad \text{and} \quad \mathbf{q} = q(x) = 1 + x^2 + x^4$$

(a) Compute $\langle \mathbf{p}, \mathbf{q} \rangle$, $\|\mathbf{p}\|$, and $\|\mathbf{q}\|$.

(b) Verify that the identities in Exercises 44 and 45 hold for the vectors $\mathbf{p}$ and $\mathbf{q}$.

T3. Let the vector space M_{33} have the standard inner product and let

$$\mathbf{u} = U = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 1 \\ 3 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

(a) Use Formula (8) to compute $\langle \mathbf{u}, \mathbf{v} \rangle$, $\|\mathbf{u}\|$, and $\|\mathbf{v}\|$.

(b) Verify that the identities in Exercises 44 and 45 hold for the vectors $\mathbf{u}$ and $\mathbf{v}$.

6.2 Angle and Orthogonality in Inner Product Spaces

In Section 3.2 we defined the notion of "angle" between vectors in R^n. In this section we will extend this idea to general vector spaces. This will enable us to extend the notion of orthogonality as well, thereby setting the groundwork for a variety of new applications.

Cauchy–Schwarz Inequality

Recall from Formula (20) of Section 3.2 that the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right) \tag{1}$$

We were assured that this formula was valid because it followed from the Cauchy–Schwarz inequality (Theorem 3.2.4) that

$$-1 \le \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \le 1 \tag{2}$$

as required for the inverse cosine to be defined. The following generalization of the Cauchy–Schwarz inequality will enable us to define the angle between two vectors in *any* real inner product space.

THEOREM 6.2.1 Cauchy–Schwarz Inequality

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in a real inner product space* V, *then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \|\mathbf{v}\| \tag{3}$$

Proof We warn you in advance that the proof presented here depends on a clever trick that is not easy to motivate.

In the case where $\mathbf{u} = \mathbf{0}$ the two sides of (3) are equal since $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\|\mathbf{u}\|$ are both zero. Thus, we need only consider the case where $\mathbf{u} \ne \mathbf{0}$. Making this assumption, let

$$a = \langle \mathbf{u}, \mathbf{u} \rangle, \quad b = 2\langle \mathbf{u}, \mathbf{v} \rangle, \quad c = \langle \mathbf{v}, \mathbf{v} \rangle$$

and let t be any real number. Since the positivity axiom states that the inner product of any vector with itself is nonnegative, it follows that

$$\begin{aligned} 0 \leq \langle t\mathbf{u} + \mathbf{v}, t\mathbf{u} + \mathbf{v}\rangle &= \langle \mathbf{u}, \mathbf{u}\rangle t^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle t + \langle \mathbf{v}, \mathbf{v}\rangle \\ &= at^2 + bt + c \end{aligned}$$

This inequality implies that the quadratic polynomial $at^2 + bt + c$ has either no real roots or a repeated real root. Therefore, its discriminant must satisfy the inequality $b^2 - 4ac \leq 0$. Expressing the coefficients a, b, and c in terms of the vectors $\mathbf{u}$ and $\mathbf{v}$ gives $4\langle \mathbf{u}, \mathbf{v}\rangle^2 - 4\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle \leq 0$ or, equivalently,

$$\langle \mathbf{u}, \mathbf{v}\rangle^2 \leq \langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle$$

Taking square roots of both sides and using the fact that $\langle \mathbf{u}, \mathbf{u}\rangle$ and $\langle \mathbf{v}, \mathbf{v}\rangle$ are nonnegative yields

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \langle \mathbf{u}, \mathbf{u}\rangle^{1/2}\langle \mathbf{v}, \mathbf{v}\rangle^{1/2} \quad \text{or equivalently} \quad |\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

which completes the proof. ◀

The following two alternative forms of the Cauchy–Schwarz inequality are useful to know:

$$\boxed{\langle \mathbf{u}, \mathbf{v}\rangle^2 \leq \langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle} \tag{4}$$

$$\boxed{\langle \mathbf{u}, \mathbf{v}\rangle^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{5}$$

The first of these formulas was obtained in the proof of Theorem 6.2.1, and the second is a variation of the first.

Angle Between Vectors

Our next goal is to define what is meant by the "angle" between vectors in a real inner product space. As a first step, we leave it as an exercise for you to use the Cauchy–Schwarz inequality to show that

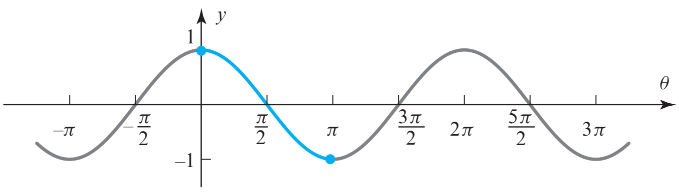
$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1 \tag{6}$$

This being the case, there is a unique angle θ in radian measure for which

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \quad \text{and} \quad 0 \leq \theta \leq \pi \tag{7}$$

(Figure 6.2.1). This enables us to *define* the ***angle θ between u and v*** to be

$$\boxed{\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)} \tag{8}$$



▶ Figure 6.2.1

EXAMPLE 1 Cosine of the Angle Between Vectors in M_{22}

Let M_{22} have the standard inner product. Find the cosine of the angle between the vectors

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

Solution We showed in Example 6 of the previous section that

$$\langle \mathbf{u}, \mathbf{v} \rangle = 16, \quad \|\mathbf{u}\| = \sqrt{30}, \quad \|\mathbf{v}\| = \sqrt{14}$$

from which it follows that

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{16}{\sqrt{30}\sqrt{14}} \approx 0.78$$

Properties of Length and Distance in General Inner Product Spaces

In Section 3.2 we used the dot product to extend the notions of length and distance to R^n, and we showed that various basic geometry theorems remained valid (see Theorems 3.2.5, 3.2.6, and 3.2.7). By making only minor adjustments to the proofs of those theorems, one can show that they remain valid in any real inner product space. For example, here is the generalization of Theorem 3.2.5 (the triangle inequalities).

$\|u\| = \sqrt{\langle u,u \rangle}$

$\equiv \|u\|^2 = \langle u,u \rangle$

THEOREM 6.2.2 *If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a real inner product space V, and if k is any scalar, then:*

(*a*) $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(*b*) $d(\mathbf{u}, \mathbf{v}) \le d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ **[Triangle inequality for distances]**

Proof (a)

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&\le \langle \mathbf{u}, \mathbf{u} \rangle + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + \langle \mathbf{v}, \mathbf{v} \rangle && \textbf{[Property of absolute value]} \\
&\le \langle \mathbf{u}, \mathbf{u} \rangle + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \langle \mathbf{v}, \mathbf{v} \rangle && \textbf{[By (3)]} \\
&= \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
\end{aligned}$$

$= \langle u, u+v \rangle + \langle v, u+v \rangle = \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$

Taking square roots gives $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

Proof (b) Identical to the proof of part (*b*) of Theorem 3.2.5.

Orthogonality

Although Example 1 is a useful mathematical exercise, there is only an occasional need to compute angles in vector spaces other than R^2 and R^3. A problem of more interest in general vector spaces is ascertaining whether the angle between vectors is $\pi/2$. You should be able to see from Formula (8) that if $\mathbf{u}$ and $\mathbf{v}$ are *nonzero* vectors, then the angle between them is $\theta = \pi/2$ if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. Accordingly, we make the following definition, which is a generalization of Definition 1 in Section 3.3 and is applicable even if one or both of the vectors is zero.

DEFINITION 1 Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V called ***orthogonal*** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Proof (b) It follows from part (*a*) and Formula (11) that

$$\begin{aligned}
d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| &= \|(\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v})\| \\
&\le \|\mathbf{u} - \mathbf{w}\| + \|\mathbf{w} - \mathbf{v}\| = d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})
\end{aligned}$$

Part (a)

As the following example shows, orthogonality depends on the inner product in the sense that for different inner products two vectors can be orthogonal with respect to one but not the other.

▶ EXAMPLE 2 Orthogonality Depends on the Inner Product

The vectors $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$ are orthogonal with respect to the Euclidean inner product on R^2 since

$1 - 1 = 0$

$$\mathbf{u} \cdot \mathbf{v} = (1)(1) + (1)(-1) = 0$$

However, they are not orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ since

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3(1)(1) + 2(1)(-1) = 1 \neq 0$$

▶ EXAMPLE 3 Orthogonal Vectors in M_{22}

If M_{22} has the inner product of Example 6 in the preceding section, then the matrices

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are orthogonal since

$$\langle U, V \rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0$$

CALCULUS REQUIRED

▶ EXAMPLE 4 Orthogonal Vectors in P_2

Let P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \left[\int_{-1}^{1} xx\,dx\right]^{1/2} = \left[\int_{-1}^{1} x^2\,dx\right]^{1/2} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left[\int_{-1}^{1} x^2x^2\,dx\right]^{1/2} = \left[\int_{-1}^{1} x^4\,dx\right]^{1/2} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} xx^2\,dx = \int_{-1}^{1} x^3\,dx = 0$$

Because $\langle \mathbf{p}, \mathbf{q} \rangle = 0$, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to the given inner product. ◀

In Theorem 3.3.3 we proved the Theorem of Pythagoras for vectors in Euclidean n-space. The following theorem extends this result to vectors in any real inner product space.

THEOREM 6.2.3 Generalized Theorem of Pythagoras

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal vectors in a real inner product space, then*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

EXAMPLE 4 Orthogonal Vectors in P_2

Let P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\, dx$$

and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$<p ; q> = 0$

$$\int_{-1}^{1} x \cdot x^2 \, dx = \int_{-1}^{1} x^3 \, dx = \frac{x^4}{4} \Big|_{-1}^{1} = \frac{1}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = 0$$

properties

Proof The orthogonality of **u** and **v** implies that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, so

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle &= \|\mathbf{u}\|^2 + \cancel{2\langle \mathbf{u}, \mathbf{v} \rangle} + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \end{aligned}$$

CALCULUS REQUIRED

EXAMPLE 5 Theorem of Pythagoras in P_2

In Example 4 we showed that $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

on P_2. It follows from Theorem 6.2.3 that

$$\|\mathbf{p}+\mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$$

Thus, from the computations in Example 4, we have

$$\|\mathbf{p}+\mathbf{q}\|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

We can check this result by direct integration:

$$\begin{aligned} \|\mathbf{p}+\mathbf{q}\|^2 = \langle \mathbf{p}+\mathbf{q}, \mathbf{p}+\mathbf{q} \rangle &= \int_{-1}^{1} (x+x^2)(x+x^2)\,dx \\ &= \int_{-1}^{1} x^2\,dx + 2\int_{-1}^{1} x^3\,dx + \int_{-1}^{1} x^4\,dx = \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15} \end{aligned}$$

Orthogonal Complements

In Section 4.8 we defined the notion of an *orthogonal complement* for subspaces of R^n, and we used that definition to establish a geometric link between the fundamental spaces of a matrix. The following definition extends that idea to general inner product spaces.

DEFINITION 2 If W is a subspace of a real inner product space V, then the set of all vectors in V that are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$.

In Theorem 4.8.6 we stated three properties of orthogonal complements in R^n. The following theorem generalizes parts (*a*) and (*b*) of that theorem to general real inner product spaces.

THEOREM 6.2.4 *If W is a subspace of a real inner product space V, then:*

(*a*) $W^{\perp}$ *is a subspace of* V.

(*b*) $W \cap W^{\perp} = \{\mathbf{0}\}$.

Proof (a) The set $W^{\perp}$ contains at least the zero vector, since $\langle \mathbf{0}, \mathbf{w} \rangle = 0$ for every vector **w** in W. Thus, it remains to show that $W^{\perp}$ is closed under addition and scalar multiplication. To do this, suppose that **u** and **v** are vectors in $W^{\perp}$, so that for every vector **w** in W we have $\langle \mathbf{u}, \mathbf{w} \rangle = 0$ and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. It follows from the additivity and homogeneity axioms of inner products that

$$\begin{aligned} \langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle = 0 + 0 = 0 \\ \langle k\mathbf{u}, \mathbf{w} \rangle &= k\langle \mathbf{u}, \mathbf{w} \rangle = k(0) = 0 \end{aligned}$$

which proves that $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ are in $W^{\perp}$.

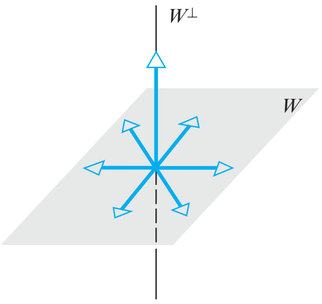
Proof (b) If $\mathbf{v}$ is any vector in both W and $W^{\perp}$, then $\mathbf{v}$ is orthogonal to itself; that is, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$. It follows from the positivity axiom for inner products that $\mathbf{v} = \mathbf{0}$. ◀

The next theorem, which we state without proof, generalizes part (c) of Theorem 4.8.6. Note, however, that this theorem applies only to finite-dimensional inner product spaces, whereas Theorem 4.8.6 does not have this restriction.

Theorem 6.2.5 implies that in a finite-dimensional inner product space orthogonal complements occur in pairs, each being orthogonal to the other (Figure 6.2.2).

THEOREM 6.2.5 *If W is a subspace of a real finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W; that is,*

$$(W^{\perp})^{\perp} = W$$



▲ Figure 6.2.2 Each vector in W is orthogonal to each vector in $W^{\perp}$ and conversely.

In our study of the fundamental spaces of a matrix in Section 4.8 we showed that the row space and null space of a matrix are orthogonal complements with respect to the Euclidean inner product on R^n (Theorem 4.8.7). The following example takes advantage of that fact.

▶ EXAMPLE 6 Basis for an Orthogonal Complement

Let W be the subspace of R^6 spanned by the vectors

$$\mathbf{w}_1 = (1, 3, -2, 0, 2, 0), \quad \mathbf{w}_2 = (2, 6, -5, -2, 4, -3),$$
$$\mathbf{w}_3 = (0, 0, 5, 10, 0, 15), \quad \mathbf{w}_4 = (2, 6, 0, 8, 4, 18)$$

Find a basis for the orthogonal complement of W.

Solution The subspace W is the same as the row space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Since the row space and null space of A are orthogonal complements, our problem reduces to finding a basis for the null space of this matrix. In Example 4 of Section 4.7 we showed that

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for this null space. Expressing these vectors in comma-delimited form (to match that of $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$), we obtain the basis vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

You may want to check that these vectors are orthogonal to $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ by computing the necessary dot products. ◀

Exercise Set 6.2

In Exercises 1–2, find the cosine of the angle between the vectors with respect to the Euclidean inner product.

1. (a) $\mathbf{u} = (1, -3),\ \mathbf{v} = (2, 4)$
   (b) $\mathbf{u} = (-1, 5, 2),\ \mathbf{v} = (2, 4, -9)$
   (c) $\mathbf{u} = (1, 0, 1, 0),\ \mathbf{v} = (-3, -3, -3, -3)$

2. (a) $\mathbf{u} = (-1, 0),\ \mathbf{v} = (3, 8)$
   (b) $\mathbf{u} = (4, 1, 8),\ \mathbf{v} = (1, 0, -3)$
   (c) $\mathbf{u} = (2, 1, 7, -1),\ \mathbf{v} = (4, 0, 0, 0)$

In Exercises 3–4, find the cosine of the angle between the vectors with respect to the standard inner product on P_2.

3. $\mathbf{p} = -1 + 5x + 2x^2,\ \ \mathbf{q} = 2 + 4x - 9x^2$

4. $\mathbf{p} = x - x^2,\ \ \mathbf{q} = 7 + 3x + 3x^2$

In Exercises 5–6, find the cosine of the angle between A and B with respect to the standard inner product on M_{22}.

5. $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix},\ \ B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix},\ \ B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

In Exercises 7–8, determine whether the vectors are orthogonal with respect to the Euclidean inner product.

7. (a) $\mathbf{u} = (-1, 3, 2),\ \mathbf{v} = (4, 2, -1)$
   (b) $\mathbf{u} = (-2, -2, -2),\ \mathbf{v} = (1, 1, 1)$
   (c) $\mathbf{u} = (a, b),\ \mathbf{v} = (-b, a)$

8. (a) $\mathbf{u} = (u_1, u_2, u_3),\ \mathbf{v} = (0, 0, 0)$
   (b) $\mathbf{u} = (-4, 6, -10, 1),\ \mathbf{v} = (2, 1, -2, 9)$
   (c) $\mathbf{u} = (a, b, c),\ \mathbf{v} = (-c, 0, a)$

In Exercises 9–10, show that the vectors are orthogonal with respect to the standard inner product on P_2.

9. $\mathbf{p} = -1 - x + 2x^2,\ \ \mathbf{q} = 2x + x^2$

10. $\mathbf{p} = 2 - 3x + x^2,\ \ \mathbf{q} = 4 + 2x - 2x^2$

In Exercises 11–12, show that the matrices are orthogonal with respect to the standard inner product on M_{22}.

11. $U = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix},\ \ V = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

12. $U = \begin{bmatrix} 5 & -1 \\ 2 & -2 \end{bmatrix},\ \ V = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$

In Exercises 13–14, show that the vectors are not orthogonal with respect to the Euclidean inner product on R^2, and then find a value of k for which the vectors are orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + ku_2v_2$.

13. $\mathbf{u} = (1, 3),\ \ \mathbf{v} = (2, -1)$

14. $\mathbf{u} = (2, -4),\ \ \mathbf{v} = (0, 3)$

15. If the vectors $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (2, -4)$ are orthogonal with respect to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2$, what must be true of the weights w_1 and w_2?

16. Let R^4 have the Euclidean inner product. Find two unit vectors that are orthogonal to all three of the vectors $\mathbf{u} = (2, 1, -4, 0)$, $\mathbf{v} = (-1, -1, 2, 2)$, and $\mathbf{w} = (3, 2, 5, 4)$.

17. Do there exist scalars k and l such that the vectors

$$\mathbf{p}_1 = 2 + kx + 6x^2, \quad \mathbf{p}_2 = l + 5x + 3x^2, \quad \mathbf{p}_3 = 1 + 2x + 3x^2$$

are mutually orthogonal with respect to the standard inner product on P_2?

18. Show that the vectors

$$\mathbf{u} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$$

are orthogonal with respect to the inner product on R^2 that is generated by the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

[See Formulas (5) and (6) of Section 6.1.]

19. Let P_2 have the evaluation inner product at the points

$$x_0 = -2, \quad x_1 = 0, \quad x_2 = 2$$

Show that the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to this inner product.

20. Let M_{22} have the standard inner product. Determine whether the matrix A is in the subspace spanned by the matrices U and V.

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 4 & 0 \\ 9 & 2 \end{bmatrix}$$

In Exercises 21–24, confirm that the Cauchy–Schwarz inequality holds for the given vectors using the stated inner product.

21. $\mathbf{u} = (1, 0, 3)$, $\mathbf{v} = (2, 1, -1)$ using the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$ in R^3.

22. $U = \begin{bmatrix} -1 & 2 \\ 6 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix}$

using the standard inner product on M_{22}.

23. $\mathbf{p} = -1 + 2x + x^2$ and $\mathbf{q} = 2 - 4x^2$ using the standard inner product on P_2.

24. The vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

with respect to the inner product in Exercise 18.

25. Let R^4 have the Euclidean inner product, and let $\mathbf{u} = (-1, 1, 0, 2)$. Determine whether the vector $\mathbf{u}$ is orthogonal to the subspace spanned by the vectors $\mathbf{w}_1 = (1, -1, 3, 0)$ and $\mathbf{w}_2 = (4, 0, 9, 2)$.

26. Let P_3 have the standard inner product, and let

$$\mathbf{p} = -1 - x + 2x^2 + 4x^3$$

Determine whether $\mathbf{p}$ is orthogonal to the subspace spanned by the polynomials $\mathbf{w}_1 = 2 - x^2 + x^3$ and $\mathbf{w}_2 = 4x - 2x^2 + 2x^3$.

In Exercises 27–28, find a basis for the orthogonal complement of the subspace of R^n spanned by the vectors.

27. $\mathbf{v}_1 = (1, 4, 5, 2)$, $\mathbf{v}_2 = (2, 1, 3, 0)$, $\mathbf{v}_3 = (-1, 3, 2, 2)$

28. $\mathbf{v}_1 = (1, 4, 5, 6, 9)$, $\mathbf{v}_2 = (3, -2, 1, 4, -1)$, $\mathbf{v}_3 = (-1, 0, -1, -2, -1)$, $\mathbf{v}_4 = (2, 3, 5, 7, 8)$

In Exercises 29–30, assume that R^n has the Euclidean inner product.

29. (a) Let W be the line in R^2 with equation $y = 2x$. Find an equation for $W^\perp$.

(b) Let W be the plane in R^3 with equation $x - 2y - 3z = 0$. Find parametric equations for $W^\perp$.

30. (a) Let W be the y-axis in an xyz-coordinate system in R^3. Describe the subspace $W^\perp$.

(b) Let W be the yz-plane of an xyz-coordinate system in R^3. Describe the subspace $W^\perp$.

31. ***(Calculus required)*** Let $C[0, 1]$ have the integral inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\,dx$$

and let $\mathbf{p} = p(x) = x$ and $\mathbf{q} = q(x) = x^2$.

(a) Find $\langle \mathbf{p}, \mathbf{q} \rangle$.

(b) Find $\|\mathbf{p}\|$ and $\|\mathbf{q}\|$.

32. (a) Find the cosine of the angle between the vectors $\mathbf{p}$ and $\mathbf{q}$ in Exercise 31.

(b) Find the distance between the vectors $\mathbf{p}$ and $\mathbf{q}$ in Exercise 31.

33. ***(Calculus required)*** Let $C[-1, 1]$ have the integral inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x)\,dx$$

and let $\mathbf{p} = p(x) = x^2 - x$ and $\mathbf{q} = q(x) = x + 1$.

(a) Find $\langle \mathbf{p}, \mathbf{q} \rangle$.

(b) Find $\|\mathbf{p}\|$ and $\|\mathbf{q}\|$.

34. (a) Find the cosine of the angle between the vectors $\mathbf{p}$ and $\mathbf{q}$ in Exercise 33.

(b) Find the distance between the vectors $\mathbf{p}$ and $\mathbf{q}$ in Exercise 33.

35. ***(Calculus required)*** Let $C[0, 1]$ have the inner product in Exercise 31.

(a) Show that the vectors

$$\mathbf{p} = p(x) = 1 \quad \text{and} \quad \mathbf{q} = q(x) = \tfrac{1}{2} - x$$

are orthogonal.

(b) Show that the vectors in part (a) satisfy the Theorem of Pythagoras.

36. ***(Calculus required)*** Let $C[-1, 1]$ have the inner product in Exercise 33.

(a) Show that the vectors

$$\mathbf{p} = p(x) = x \quad \text{and} \quad \mathbf{q} = q(x) = x^2 - 1$$

are orthogonal.

(b) Show that the vectors in part (a) satisfy the Theorem of Pythagoras.

37. Let V be an inner product space. Show that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors in V, then $\|\mathbf{u} - \mathbf{v}\| = \sqrt{2}$.

38. Let V be an inner product space. Show that if $\mathbf{w}$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$, then it is orthogonal to $k_1\mathbf{u}_1 + k_2\mathbf{u}_2$ for all scalars k_1 and k_2. Interpret this result geometrically in the case where V is R^3 with the Euclidean inner product.

39. ***(Calculus required)*** Let $C[0, \pi]$ have the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^\pi f(x)g(x)\,dx$$

and let $\mathbf{f}_n = \cos nx$ $(n = 0, 1, 2, \ldots)$. Show that if $k \neq l$, then $\mathbf{f}_k$ and $\mathbf{f}_l$ are orthogonal vectors.

40. As illustrated in the accompanying figure, the vectors $\mathbf{u} = (1, \sqrt{3})$ and $\mathbf{v} = (-1, \sqrt{3})$ have norm 2 and an angle of 60° between them relative to the Euclidean inner product. Find a weighted Euclidean inner product with respect to which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors.

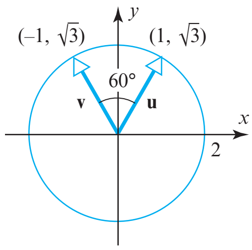


Figure Ex-40

Working with Proofs

41. Let V be an inner product space. Prove that if $\mathbf{w}$ is orthogonal to each of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$, then it is orthogonal to every vector in span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$.

42. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a basis for an inner product space V. Prove that the zero vector is the only vector in V that is orthogonal to all of the basis vectors.

43. Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be a basis for a subspace W of V. Prove that $W^\perp$ consists of all vectors in V that are orthogonal to every basis vector.

44. Prove the following generalization of Theorem 6.2.3: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are pairwise orthogonal vectors in an inner product space V, then

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_r\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \cdots + \|\mathbf{v}_r\|^2$$

45. Prove: If $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices and A is an $n \times n$ matrix, then

$$(\mathbf{v}^T A^T A\mathbf{u})^2 \le (\mathbf{u}^T A^T A\mathbf{u})(\mathbf{v}^T A^T A\mathbf{v})$$

46. Use the Cauchy–Schwarz inequality to prove that for all real values of a, b, and θ,

$$(a\cos\theta + b\sin\theta)^2 \le a^2 + b^2$$

47. Prove: If $w_1, w_2, \ldots, w_n$ are positive real numbers, and if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are any two vectors in R^n, then

$$|w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n|$$
$$\le (w_1u_1^2 + w_2u_2^2 + \cdots + w_nu_n^2)^{1/2}(w_1v_1^2 + w_2v_2^2 + \cdots + w_nv_n^2)^{1/2}$$

48. Prove that equality holds in the Cauchy–Schwarz inequality if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

49. ***(Calculus required)*** Let $f(x)$ and $g(x)$ be continuous functions on $[0, 1]$. Prove:

(a) $$\left[\int_0^1 f(x)g(x)\,dx\right]^2 \le \left[\int_0^1 f^2(x)\,dx\right]\left[\int_0^1 g^2(x)\,dx\right]$$

(b) $$\left[\int_0^1 [f(x)+g(x)]^2\,dx\right]^{1/2} \le \left[\int_0^1 f^2(x)\,dx\right]^{1/2} + \left[\int_0^1 g^2(x)\,dx\right]^{1/2}$$

[*Hint:* Use the Cauchy–Schwarz inequality.]

50. Prove that Formula (4) holds for all nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in a real inner product space V.

51. Let $T_A\colon R^2 \to R^2$ be multiplication by

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and let $\mathbf{x} = (1, 1)$.

(a) Assuming that R^2 has the Euclidean inner product, find all vectors $\mathbf{v}$ in R^2 such that $\langle \mathbf{x}, \mathbf{v}\rangle = \langle T_A(\mathbf{x}), T_A(\mathbf{v})\rangle$.

(b) Assuming that R^2 has the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v}\rangle = 2u_1v_1 + 3u_2v_2$, find all vectors $\mathbf{v}$ in R^2 such that $\langle \mathbf{x}, \mathbf{v}\rangle = \langle T_A(\mathbf{x}), T_A(\mathbf{v})\rangle$.

52. Let $T\colon P_2 \to P_2$ be the linear transformation defined by

$$T(a + bx + cx^2) = 3a - cx^2$$

and let $\mathbf{p} = 1 + x$.

(a) Assuming that P_2 has the standard inner product, find all vectors $\mathbf{q}$ in P_2 such that $\langle \mathbf{p}, \mathbf{q}\rangle = \langle T(\mathbf{p}), T(\mathbf{q})\rangle$.

(b) Assuming that P_2 has the evaluation inner product at the points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, find all vectors $\mathbf{q}$ in P_2 such that $\langle \mathbf{p}, \mathbf{q}\rangle = \langle T(\mathbf{p}), T(\mathbf{q})\rangle$.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If $\mathbf{u}$ is orthogonal to every vector of a subspace W, then $\mathbf{u} = \mathbf{0}$.

(b) If $\mathbf{u}$ is a vector in both W and $W^\perp$, then $\mathbf{u} = \mathbf{0}$.

(c) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $W^\perp$, then $\mathbf{u} + \mathbf{v}$ is in $W^\perp$.

(d) If $\mathbf{u}$ is a vector in $W^\perp$ and k is a real number, then $k\mathbf{u}$ is in $W^\perp$.

(e) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $|\langle \mathbf{u}, \mathbf{v}\rangle| = \|\mathbf{u}\|\|\mathbf{v}\|$.

(f) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

Working with Technology

T1. (a) We know that the row space and null space of a matrix are orthogonal complements relative to the Euclidean inner product. Confirm this fact for the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 4 & -3 & 1 & 3 \\ 3 & -2 & 3 & 4 \\ 4 & -1 & 15 & 17 \\ 7 & -6 & -7 & 0 \end{bmatrix}$$

(b) Find a basis for the orthogonal complement of the column space of A.

T2. In each part, confirm that the vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy the Cauchy–Schwarz inequality relative to the stated inner product.

(a) M_{44} with the standard inner product.

$$\mathbf{u} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 4 & -3 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 & 2 & 1 & 3 \\ 3 & -1 & 0 & 1 \\ 1 & 0 & 0 & -2 \\ -3 & 1 & 2 & 0 \end{bmatrix}$$

(b) R^4 with the weighted Euclidean inner product with weights $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{4}$, $w_3 = \frac{1}{8}$, $w_4 = \frac{1}{8}$.

$$\mathbf{u} = (1, -2, 2, 1) \quad \text{and} \quad \mathbf{v} = (0, -3, 3, -2)$$