39. Prove that the intersection of any two distinct eigenspaces of a matrix A is $\{\mathbf{0}\}$.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If A is a square matrix and $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero scalar λ, then $\mathbf{x}$ is an eigenvector of A.

(b) If λ is an eigenvalue of a matrix A, then the linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) If the characteristic polynomial of a matrix A is $p(\lambda) = \lambda^2 + 1$, then A is invertible.

(d) If λ is an eigenvalue of a matrix A, then the eigenspace of A corresponding to λ is the set of eigenvectors of A corresponding to λ.

(e) The eigenvalues of a matrix A are the same as the eigenvalues of the reduced row echelon form of A.

(f) If 0 is an eigenvalue of a matrix A, then the set of columns of A is linearly independent.

$d(\lambda I - A) = 0$

Working with Technology

T1. For the given matrix A, find the characteristic polynomial and the eigenvalues, and then use the method of Example 7 to find bases for the eigenspaces.

$$A = \begin{bmatrix} -8 & 33 & 38 & 173 & -30 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & -5 & -25 & 1 \\ 0 & 0 & 1 & 5 & 0 \\ 4 & -16 & -19 & -86 & 15 \end{bmatrix}$$

T2. The Cayley–Hamilton Theorem states that every square matrix satisfies its characteristic equation; that is, if A is an $n \times n$ matrix whose characteristic equation is

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

then $A^n + c_1A^{n-1} + \cdots + c_n = 0$.

(a) Verify the Cayley–Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

(b) Use the result in Exercise 28 to prove the Cayley–Hamilton Theorem for 2×2 matrices.

$(\lambda I - A)x = \mathbf{0}$

$\lambda = \lambda_0$

5.2 Diagonalization

In this section we will be concerned with the problem of finding a basis for R^n that consists of eigenvectors of an $n \times n$ matrix A. Such bases can be used to study geometric properties of A and to simplify various numerical computations. These bases are also of physical significance in a wide variety of applications, some of which will be considered later in this text.

The Matrix Diagonalization Problem

Products of the form $P^{-1}AP$ in which A and P are $n \times n$ matrices and P is invertible will be our main topic of study in this section. There are various ways to think about such products, one of which is to view them as transformations

$$A \to P^{-1}AP = B$$

in which the matrix A is mapped into the matrix $P^{-1}AP$. These are called ***similarity transformations***. Such transformations are important because they preserve many properties of the matrix A. For example, if we let $B = P^{-1}AP$, then A and B have the same determinant since

$$\boxed{\det(B)} = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$
$$= \frac{1}{\cancel{\det(P)}}\det(A)\cancel{\det(P)} = \boxed{\det(A)}$$

In general, any property that is preserved by a similarity transformation is called a ***similarity invariant*** and is said to be ***invariant under similarity***. Table 1 lists the most important similarity invariants. The proofs of some of these are given as exercises.

Table 1 Similarity Invariants

A $B = P^{-1}AP$

Property	**Description**
Determinant	A and $P^{-1}AP$ have the same determinant.
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible.
Rank	A and $P^{-1}AP$ have the same rank.
Nullity	A and $P^{-1}AP$ have the same nullity.
Trace	A and $P^{-1}AP$ have the same trace.
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial.
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues.
Eigenspace dimension	If λ is an eigenvalue of A (and hence of $P^{-1}AP$) then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

We will find the following terminology useful in our study of similarity transformations.

$A \sim B$

$n \times n$

DEFINITION 1 If A and B are square matrices, then we say that ***B is similar to A*** if there is an invertible matrix P such that $B = P^{-1}AP$.

$\exists P$ (invertiable) $B = P^{-1}AP$

Note that if B is similar to A, then it is also true that A is similar to B since we can express A as $A = Q^{-1}BQ$ by taking $Q = P^{-1}$. This being the case, we will usually say that A and B are ***similar matrices*** if either is similar to the other.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A \sim B$

$B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$

Because diagonal matrices have such a simple form, it is natural to inquire whether a given $n \times n$ matrix A is similar to a matrix of this type. Should this turn out to be the case, and should we be able to actually find a diagonal matrix D that is similar to A, then we would be able to ascertain many of the similarity invariant properties of A directly from the diagonal entries of D. For example, the diagonal entries of D will be the eigenvalues of A (Theorem 5.1.2), and the product of the diagonal entries of D will be the determinant of A (Theorem 2.1.2). This leads us to introduce the following terminology.

$P = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \Rightarrow \exists P^{-1}$

$A \sim D$

DEFINITION 2 A square matrix A is said to be ***diagonalizable*** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to ***diagonalize*** A.

$P^{-1}AP = B$

The following theorem and the ideas used in its proof will provide us with a roadmap for devising a technique for determining whether a matrix is diagonalizable and, if so, for finding a matrix P that will perform the diagonalization.

Part (b) of Theorem 5.2.1 is equivalent to saying that there is a *basis* for R^n consisting of eigenvectors of A. Why?

THEOREM 5.2.1 *If A is an $n \times n$ matrix, the following statements are equivalent.*

(a) *A is diagonalizable.*

(b) *A has n linearly independent eigenvectors.*

Proof (a) ⇒ (b) Since A is assumed to be diagonalizable, it follows that there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ or, equivalently,

$$AP = PD \tag{1}$$

If we denote the column vectors of P by $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, and if we assume that the diagonal entries of D are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then by Formula (6) of Section 1.3 the left side of (1) can be expressed as

$$AP = A[\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n] = [A\mathbf{p}_1 \quad A\mathbf{p}_2 \quad \cdots \quad A\mathbf{p}_n]$$

and, as noted in the comment following Example 1 of Section 1.7, the right side of (1) can be expressed as

$$PD = [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n]$$

Thus, it follows from (1) that

$$A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, \quad A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \ldots, \quad A\mathbf{p}_n = \lambda_n\mathbf{p}_n \tag{2}$$

Since P is invertible, we know from Theorem 5.1.5 that its column vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are linearly independent (and hence nonzero). Thus, it follows from (2) that these n column vectors are eigenvectors of A.

Proof (b) ⇒ (a) Assume that A has n linearly independent eigenvectors, $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, and that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues. If we let

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$$

and if we let D be the diagonal matrix that has $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, then

$$\begin{aligned} AP &= A[\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n] = [A\mathbf{p}_1 \quad A\mathbf{p}_2 \quad \cdots \quad A\mathbf{p}_n] \\ &= [\lambda_1\mathbf{p}_1 \quad \lambda_2\mathbf{p}_2 \quad \cdots \quad \lambda_n\mathbf{p}_n] = PD \end{aligned}$$

Since the column vectors of P are linearly independent, it follows from Theorem 5.1.5 that P is invertible, so that this last equation can be rewritten as $P^{-1}AP = D$, which shows that A is diagonalizable. ◀

Whereas Theorem 5.2.1 tells us that we need to find n linearly independent eigenvectors to diagonalize a matrix, the following theorem tells us where such vectors might be found. Part (a) is proved at the end of this section, and part (b) is an immediate consequence of part (a) and Theorem 5.2.1 (why?).

THEOREM 5.2.2

(a) *If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix A, and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are corresponding eigenvectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set.*

(b) *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

Remark Part (a) of Theorem 5.2.2 is a special case of a more general result: Specifically, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues, and if $S_1, S_2, \ldots, S_k$ are corresponding sets of linearly independent eigenvectors, then the *union* of these sets is linearly independent.

Procedure for Diagonalizing a Matrix

Theorem 5.2.1 guarantees that an $n \times n$ matrix A with n linearly independent eigenvectors is diagonalizable, and the proof of that theorem together with Theorem 5.2.2 suggests the following procedure for diagonalizing A.

A Procedure for Diagonalizing an $n \times n$ Matrix

Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n, then it is not.

Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$ whose column vectors are the n basis vectors you obtained in Step 1.

Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that correspond to the successive columns of P.

EXAMPLE 1 Finding a Matrix P That Diagonalizes a Matrix A

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution In Example 7 of the preceding section we found the characteristic equation of A to be

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and we found the following bases for the eigenspaces:

$$\lambda = 2: \quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad \lambda = 1: \quad \mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes A. As a check, you should verify that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general, there is no preferred order for the columns of P. Since the ith diagonal entry of $P^{-1}AP$ is an eigenvalue for the ith column vector of P, changing the order of the columns of P just changes the order of the eigenvalues on the diagonal of $P^{-1}AP$. Thus, had we written

$$P = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

EXAMPLE 1 Finding a Matrix P That Diagonalizes a Matrix A

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

① $\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0 \Rightarrow (\lambda-2)\begin{vmatrix} \lambda & 2 \\ -1 & \lambda-3 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)[\lambda(\lambda-3)+2] = 0 \Rightarrow (\lambda-2)(\lambda^2-3\lambda+2) = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-1) = 0$$

$$\Rightarrow \lambda-2=0 \quad \text{or} \quad \lambda-2=0 \quad \text{or} \quad \lambda-1=0$$

$$\boxed{\lambda=2} \qquad \boxed{\lambda=2} \qquad \boxed{\lambda=1}$$

To find eigenvectors

- $\boxed{\lambda=2}$

$$(\lambda I - A)x = 0 \Rightarrow \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 + 2x_3 = 0 \Rightarrow x_1 + x_3 = 0$$

$$-x_1 - x_3 = 0 \Rightarrow x_1 + x_3 = 0$$

let $\boxed{x_2 = t}$, $t \in \mathbb{R}$

let $\boxed{x_3 = s}$, $s \in \mathbb{R}$ $\Rightarrow \boxed{x_1 = -s}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = t\overset{v_1}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + s\overset{v_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\boxed{\lambda=1}$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & +1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & +1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0 \Rightarrow \boxed{x_1 = -2x_3}$

$-x_2 + x_3 = 0 \Rightarrow \boxed{x_3 = x_2}$

let $\boxed{x_3 = t}$, $t \in \mathbb{R}$

$\Rightarrow \boxed{x_1 = -2t}$, $\boxed{x_2 = t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t \overset{V_3}{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$$

$$S = \{V_1, V_2, V_3\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\therefore$ A is diagonalizable because we have 3 distinct lin- inde- eigenvector

$$P = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2, 2, 1

in the preceding example, we would have obtained

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

▶ EXAMPLE 2 A Matrix That Is Not Diagonalizable

Show that the following matrix is not diagonalizable:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)\left((\lambda - 2)^2 - 0\right) = (\lambda-1)(\lambda-2)^2$$

$\lambda = 1$, $\lambda = 2$, $\lambda = 2$

so the characteristic equation is

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and the distinct eigenvalues of A are $\lambda = 1$ and $\lambda = 2$. We leave it for you to show that bases for the eigenspaces are

$$\lambda = 1: \quad \mathbf{p}_1 = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}; \qquad \lambda = 2: \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{\lambda = 1}$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_2 = 0 \Rightarrow \boxed{x_1 = -x_2}$

$3x_1 - 5x_2 - x_3 = 0$

Let $\boxed{x_2 = t} \Rightarrow \boxed{x_1 = -t}$

$\Rightarrow x_3 = -3t - 5t = -8t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -8t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ -8 \end{bmatrix}$$

$\boxed{\lambda = 2}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{x_1 = 0}$

$x_1 = 0$

$3x_1 - 5x_2 = 0 \Rightarrow \boxed{x_2 = 0}$

Let $\boxed{x_3 = t}$ $t \in \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since A is a 3×3 matrix and there are only two basis vectors in total, A is not diagonalizable.

Alternative Solution If you are concerned only in determining whether a matrix is diagonalizable and not with actually finding a diagonalizing matrix P, then it is not necessary to compute bases for the eigenspaces—it suffices to find the dimensions of the eigenspaces. For this example, the eigenspace corresponding to $\lambda = 1$ is the solution space of the system

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the coefficient matrix has rank 2 (verify), the nullity of this matrix is 1 by Theorem 4.8.2, and hence the eigenspace corresponding to $\lambda = 1$ is one-dimensional.

The eigenspace corresponding to $\lambda = 2$ is the solution space of the system

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This coefficient matrix also has rank 2 and nullity 1 (verify), so the eigenspace corresponding to $\lambda = 2$ is also one-dimensional. Since the eigenspaces produce a total of two basis vectors, and since three are needed, the matrix A is not diagonalizable.

EXAMPLE 3 Recognizing Diagonalizability

We saw in Example 3 of the preceding section that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

has three distinct eigenvalues: $\lambda = 4$, $\lambda = 2 + \sqrt{3}$, and $\lambda = 2 - \sqrt{3}$. Therefore, A is diagonalizable and

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$$

for some invertible matrix P. If needed, the matrix P can be found using the method shown in Example 1 of this section.

EXAMPLE 4 Diagonalizability of Triangular Matrices

From Theorem 5.1.2, the eigenvalues of a triangular matrix are the entries on its main diagonal. Thus, a triangular matrix with distinct entries on the main diagonal is diagonalizable. For example,

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

is a diagonalizable matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = -2$. ◀

Eigenvalues of Powers of a Matrix

$Ax = \lambda x$

λ eigenvalue

Since there are many applications in which it is necessary to compute high powers of a square matrix A, we will now turn our attention to that important problem. As we will see, the most efficient way to compute A^k, particularly for large values of k, is to first diagonalize A. But because diagonalizing a matrix A involves finding its eigenvalues and eigenvectors, we will need to know how these quantities are related to those of A^k. As an illustration, suppose that λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector. Then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$$

which shows not only that λ^2 is a eigenvalue of A^2 but that $\mathbf{x}$ is a corresponding eigenvector. In general, we have the following result.

Note that diagonalizability is not a requirement in Theorem 5.2.3.

THEOREM 5.2.3 *If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.*

EXAMPLE 5 Eigenvalues and Eigenvectors of Matrix Powers

In Example 2 we found the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

$\lambda = 2$

$\lambda = 1$

$v_1|_{\lambda=2} = \begin{bmatrix} -1 \\ 1 \\ -8 \end{bmatrix}$ $\quad v_2|_{\lambda=1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Do the same for A^7.

eigenvalues of A^7 are $\underbrace{2^7}_{=128}$ & $\underbrace{1^7}_{=1}$

eigenvectors of A^7 are v_1 corresponding to $\lambda = 128$ is $\begin{bmatrix} -1 \\ 1 \\ -8 \end{bmatrix}$

and v_2 corresponding to $\lambda = 1$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Solution We know from Example 2 that the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$, so the eigenvalues of A^7 are $\lambda = 1^7 = 1$ and $\lambda = 2^7 = 128$. The eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ obtained in Example 1 corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 2$ of A are also the eigenvectors corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 128$ of A^7. ◀

Computing Powers of a Matrix

The problem of computing powers of a matrix is greatly simplified when the matrix is diagonalizable. To see why this is so, suppose that A is a diagonalizable $n \times n$ matrix, that P diagonalizes A, and that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$$

Squaring both sides of this equation yields

$$(P^{-1}AP)^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = D^2$$

We can rewrite the left side of this equation as

$$(P^{-1}AP)^2 = P^{-1}AP\,P^{-1}AP = P^{-1}AIAP = P^{-1}A^2P$$

$I \quad I$

$P P^{-1} A^2 P P^{-1} D^2 P^{-1}$

$A^2 = P D^2 P^{-1}$

from which we obtain the relationship $P^{-1}A^2P = D^2$. More generally, if k is a positive integer, then a similar computation will show that

$$P^{-1}A^kP = D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

which we can rewrite as

$$A^k = PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} P^{-1} \tag{3}$$

Formula (3) reveals that raising a diagonalizable matrix A to a positive integer power has the effect of raising its eigenvalues to that power.

▶ **EXAMPLE 6 Powers of a Matrix**

Use (3) to find A^{13}, where

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution We showed in Example 1 that the matrix A is diagonalized by

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and that

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}: \left[\begin{array}{ccc|ccc} 0 & -1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[-1R3+R1]{-2R3+R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \qquad \underbrace{\begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{matrix}}_{P^{-1}}$$

$$\therefore P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

$\therefore P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}$

P, P^{-1}, D

$A^{13} = P D^{13} P^{-1}$

$= P \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{bmatrix} P^{-1}$

Thus, it follows from (3) that

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$ ◀

Remark With the method in the preceding example, most of the work is in diagonalizing A. Once that work is done, it can be used to compute any power of A. Thus, to compute A^{1000} we need only change the exponents from 13 to 1000 in (4).

Geometric and Algebraic Multiplicity

Theorem 5.2.2(b) does not completely settle the diagonalizability question since it only guarantees that a square matrix with n distinct eigenvalues is diagonalizable; it does not preclude the possibility that there may exist diagonalizable matrices with fewer than n distinct eigenvalues. The following example shows that this is indeed the case.

▶ EXAMPLE 7 The Converse of Theorem 5.2.2(b) Is False

Consider the matrices

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows from Theorem 5.1.2 that both of these matrices have only one distinct eigenvalue, namely $\lambda = 1$, and hence only one eigenspace. We leave it as an exercise for you to solve the characteristic equations

$$(\lambda I - I)\mathbf{x} = \mathbf{0} \quad \text{and} \quad (\lambda I - J)\mathbf{x} = \mathbf{0}$$

with $\lambda = 1$ and show that for I the eigenspace is three-dimensional (all of R^3) and for J it is one-dimensional, consisting of all scalar multiples of

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This shows that the converse of Theorem 5.2.2(b) is false, since we have produced two 3×3 matrices with fewer than three distinct eigenvalues, one of which is diagonalizable and the other of which is not. ◀

A full excursion into the study of diagonalizability is left for more advanced courses, but we will touch on one theorem that is important for a fuller understanding of diagonalizability. It can be proved that if λ_0 is an eigenvalue of A, then the dimension of the eigenspace corresponding to λ_0 cannot exceed the number of times that $\lambda - \lambda_0$ appears as a factor of the characteristic polynomial of A. For example, in Examples 1 and 2 the characteristic polynomial is

$$(\lambda - 1)(\lambda - 2)^2$$

Thus, the eigenspace corresponding to $\lambda = 1$ is at most (hence exactly) one-dimensional, and the eigenspace corresponding to $\lambda = 2$ is at most two-dimensional. In Example 1 the eigenspace corresponding to $\lambda = 2$ actually had dimension 2, resulting in diagonalizability, but in Example 2 the eigenspace corresponding to $\lambda = 2$ had only dimension 1, resulting in nondiagonalizability.

There is some terminology that is related to these ideas. If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the ***geometric multiplicity*** of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the ***algebraic multiplicity*** of λ_0. The following theorem, which we state without proof, summarizes the preceding discussion.

THEOREM 5.2.4 Geometric and Algebraic Multiplicity

If A is a square matrix, then:

(*a*) *For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.*

(*b*) *A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.*

We will complete this section with an optional proof of Theorem 5.2.2(*a*).

OPTIONAL

Proof of Theorem 5.2.2(a) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. We will assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent and obtain a contradiction. We can then conclude that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Since an eigenvector is nonzero by definition, $\{\mathbf{v}_1\}$ is linearly independent. Let r be the largest integer such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent. Since we are assuming that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent, r satisfies $1 \le r < k$. Moreover, by the definition of r, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{r+1}\}$ is linearly dependent. Thus, there are scalars $c_1, c_2, \ldots, c_{r+1}$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{r+1}\mathbf{v}_{r+1} = \mathbf{0} \tag{5}$$

Multiplying both sides of (5) by A and using the fact that

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \ldots, \quad A\mathbf{v}_{r+1} = \lambda_{r+1}\mathbf{v}_{r+1}$$

we obtain

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_{r+1}\lambda_{r+1}\mathbf{v}_{r+1} = \mathbf{0} \tag{6}$$

If we now multiply both sides of (5) by λ_{r+1} and subtract the resulting equation from (6) we obtain

$$c_1(\lambda_1 - \lambda_{r+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{r+1})\mathbf{v}_2 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{v}_r = \mathbf{0}$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set, this equation implies that

$$c_1(\lambda_1 - \lambda_{r+1}) = c_2(\lambda_2 - \lambda_{r+1}) = \cdots = c_r(\lambda_r - \lambda_{r+1}) = 0$$

and since $\lambda_1, \lambda_2, \ldots, \lambda_{r+1}$ are assumed to be distinct, it follows that

$$c_1 = c_2 = \cdots = c_r = 0 \tag{7}$$

Substituting these values in (5) yields

$$c_{r+1}\mathbf{v}_{r+1} = \mathbf{0}$$

Since the eigenvector $\mathbf{v}_{r+1}$ is nonzero, it follows that

$$c_{r+1} = 0 \tag{8}$$

But equations (7) and (8) contradict the fact that $c_1, c_2, \ldots, c_{r+1}$ are not all zero so the proof is complete. ◀

Exercise Set 5.2

▶ In Exercises 1–4, show that A and B are not similar matrices. ◀

1. $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

2. $A = \begin{bmatrix} 4 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

▶ In Exercises 5–8, find a matrix P that diagonalizes A, and check your work by computing $P^{-1}AP$. ◀

5. $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

6. $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$

7. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

9. Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

(c) Is A diagonalizable? Justify your conclusion.

10. Follow the directions in Exercise 9 for the matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

▶ In Exercises 11–14, find the geometric and algebraic multiplicity of each eigenvalue of the matrix A, and determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A, and find $P^{-1}AP$. ◀

11. $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

12. $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

14. $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

▶ In each part of Exercises 15–16, the characteristic equation of a matrix A is given. Find the size of the matrix and the possible dimensions of its eigenspaces. ◀

15. (a) $(\lambda - 1)(\lambda + 3)(\lambda - 5) = 0$

(b) $\lambda^2(\lambda - 1)(\lambda - 2)^3 = 0$

16. (a) $\lambda^3(\lambda^2 - 5\lambda - 6) = 0$

(b) $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

▶ In Exercises 17–18, use the method of Example 6 to compute the matrix A^{10}. ◀

17. $A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

19. Let

$$A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$$

Confirm that P diagonalizes A, and then compute A^{11}.

20. Let

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Confirm that P diagonalizes A, and then compute each of the following powers of A.

(a) A^{1000} (b) A^{-1000} (c) A^{2301} (d) A^{-2301}

21. Find A^n if n is a positive integer and

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

22. Show that the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are similar.

23. We know from Table 1 that similar matrices have the same rank. Show that the converse is false by showing that the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

have the same rank but are not similar. [*Suggestion:* If they were similar, then there would be an invertible 2×2 matrix P for which $AP = PB$. Show that there is no such matrix.]

24. We know from Table 1 that similar matrices have the same eigenvalues. Use the method of Exercise 23 to show that the converse is false by showing that the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

have the same eigenvalues but are not similar.

25. If A, B, and C are $n \times n$ matrices such that A is similar to B and B is similar to C, do you think that A must be similar to C? Justify your answer.

26. (a) Is it possible for an $n \times n$ matrix to be similar to itself? Justify your answer.

(b) What can you say about an $n \times n$ matrix that is similar to $0_{n \times n}$? Justify your answer.

(c) Is is possible for a nonsingular matrix to be similar to a singular matrix? Justify your answer.

27. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. In each part, answer the question and explain your reasoning.

(a) What can you say about the dimensions of the eigenspaces of A?

(b) What can you say about the dimensions of the eigenspaces if you know that A is diagonalizable?

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of eigenvectors of A, all of which correspond to the same eigenvalue of A, what can you say about that eigenvalue?

28. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that

(a) A is diagonalizable if $(a - d)^2 + 4bc > 0$.

(b) A is not diagonalizable if $(a - d)^2 + 4bc < 0$.

[*Hint:* See Exercise 29 of Section 5.1.]

29. In the case where the matrix A in Exercise 28 is diagonalizable, find a matrix P that diagonalizes A. [*Hint:* See Exercise 30 of Section 5.1.]

▶ In Exercises 30–33, find the standard matrix A for the given linear operator, and determine whether that matrix is diagonalizable. If diagonalizable, find a matrix P that diagonalizes A. ◀

30. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

31. $T(x_1, x_2) = (-x_2, -x_1)$

32. $T(x_1, x_2, x_3) = (8x_1 + 3x_2 - 4x_3, -3x_1 + x_2 + 3x_3, 4x_1 + 3x_2)$

33. $T(x_1, x_2, x_3) = (3x_1, x_2, x_1 - x_2)$

34. If P is a fixed $n \times n$ matrix, then the similarity transformation

$$A \to P^{-1}AP$$

can be viewed as an operator $S_P(A) = P^{-1}AP$ on the vector space M_{nn} of $n \times n$ matrices.

(a) Show that S_P is a linear operator.

(b) Find the kernel of S_P.

(c) Find the rank of S_P.

Working with Proofs

35. Prove that similar matrices have the same rank and nullity.

36. Prove that similar matrices have the same trace.

37. Prove that if A is diagonalizable, then so is A^k for every positive integer k.

38. We know from Table 1 that similar matrices, A and B, have the same eigenvalues. However, it is not true that those eigenvalues have the same corresponding eigenvectors for the two matrices. Prove that if $B = P^{-1}AP$, and $\mathbf{v}$ is an eigenvector of B corresponding to the eigenvalue λ, then $P\mathbf{v}$ is the eigenvector of A corresponding to λ.

39. Let A be an $n \times n$ matrix, and let $q(A)$ be the matrix

$$q(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_n$$

(a) Prove that if $B = P^{-1}AP$, then $q(B) = P^{-1}q(A)P$.

(b) Prove that if A is diagonalizable, then so is $q(A)$.

40. Prove that if A is a diagonalizable matrix, then the rank of A is the number of nonzero eigenvalues of A.

41. This problem will lead you through a proof of the fact that the algebraic multiplicity of an eigenvalue of an $n \times n$ matrix A is greater than or equal to the geometric multiplicity. For this purpose, assume that λ_0 is an eigenvalue with geometric multiplicity k.

(a) Prove that there is a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for R^n in which the first k vectors of B form a basis for the eigenspace corresponding to λ_0.

(b) Let P be the matrix having the vectors in B as columns. Prove that the product AP can be expressed as

$$AP = P\begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

[*Hint:* Compare the first k column vectors on both sides.]

(c) Use the result in part (b) to prove that A is similar to

$$C = \begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

and hence that A and C have the same characteristic polynomial.

(d) By considering $\det(\lambda I - C)$, prove that the characteristic polynomial of C (and hence A) contains the factor $(\lambda - \lambda_0)$ at least k times, thereby proving that the algebraic multiplicity of λ_0 is greater than or equal to the geometric multiplicity k.

True-False Exercises

TF. In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) An $n \times n$ matrix with fewer than n distinct eigenvalues is not diagonalizable.

(b) An $n \times n$ matrix with fewer than n linearly independent eigenvectors is not diagonalizable.

(c) If A and B are similar $n \times n$ matrices, then there exists an invertible $n \times n$ matrix P such that $PA = BP$.

(d) If A is diagonalizable, then there is a unique matrix P such that $P^{-1}AP$ is diagonal.

(e) If A is diagonalizable and invertible, then A^{-1} is diagonalizable.

(f) If A is diagonalizable, then A^T is diagonalizable.

(g) If there is a basis for R^n consisting of eigenvectors of an $n \times n$ matrix A, then A is diagonalizable.

(h) If every eigenvalue of a matrix A has algebraic multiplicity 1, then A is diagonalizable.

(i) If 0 is an eigenvalue of a matrix A, then A^2 is singular.

Working with Technology

T1. Generate a random 4×4 matrix A and an invertible 4×4 matrix P and then confirm, as stated in Table 1, that $P^{-1}AP$ and A have the same

(a) determinant.

(b) rank.

(c) nullity.

(d) trace.

(e) characteristic polynomial.

(f) eigenvalues.

T2. (a) Use Theorem 5.2.1 to show that the following matrix is diagonalizable.

$$A = \begin{bmatrix} -13 & -60 & -60 \\ 10 & 42 & 40 \\ -5 & -20 & -18 \end{bmatrix}$$

(b) Find a matrix P that diagonalizes A.

(c) Use the method of Example 6 to compute A^{10}, and check your result by computing A^{10} directly.

T3. Use Theorem 5.2.1 to show that the following matrix is not diagonalizable.

$$A = \begin{bmatrix} -10 & 11 & -6 \\ -15 & 16 & -10 \\ -3 & 3 & -2 \end{bmatrix}$$

5.3 Complex Vector Spaces

Because the characteristic equation of any square matrix can have complex solutions, the notions of complex eigenvalues and eigenvectors arise naturally, even within the context of matrices with real entries. In this section we will discuss this idea and apply our results to study symmetric matrices in more detail. A review of the essentials of complex numbers appears in the back of this text.

Review of Complex Numbers

Recall that if $z = a + bi$ is a complex number, then:

- $\text{Re}(z) = a$ and $\text{Im}(z) = b$ are called the ***real part*** of z and the ***imaginary part*** of z, respectively,
- $|z| = \sqrt{a^2 + b^2}$ is called the ***modulus*** (or ***absolute value***) of z,
- $\overline{z} = a - bi$ is called the ***complex conjugate*** of z,