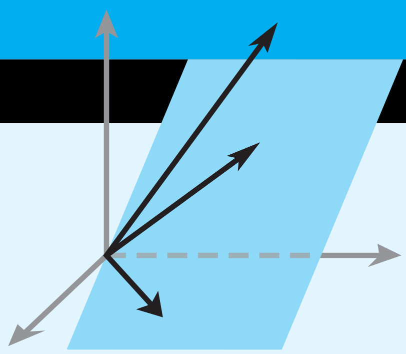


CHAPTER 5

Eigenvalues and Eigenvectors

CHAPTER CONTENTS

INTRODUCTION

In this chapter we will focus on classes of scalars and vectors known as "eigenvalues" and "eigenvectors," terms derived from the German word *eigen*, meaning "own," "peculiar to," "characteristic," or "individual." The underlying idea first appeared in the study of rotational motion but was later used to classify various kinds of surfaces and to describe solutions of certain differential equations. In the early 1900s it was applied to matrices and matrix transformations, and today it has applications in such diverse fields as computer graphics, mechanical vibrations, heat flow, population dynamics, quantum mechanics, and economics, to name just a few.

5.1 Eigenvalues and Eigenvectors

In this section we will define the notions of "eigenvalue" and "eigenvector" and discuss some of their basic properties.

Definition of Eigenvalue and Eigenvector

We begin with the main definition in this section.

DEFINITION 1 If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an ***eigenvector*** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

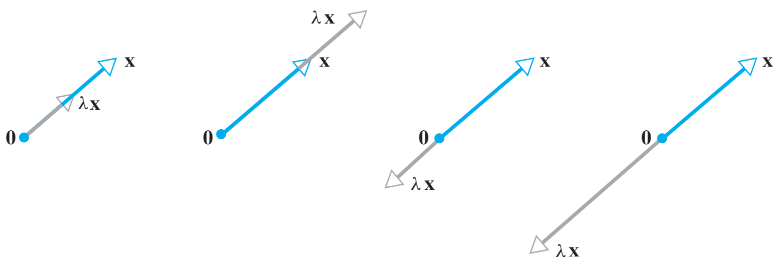
$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** of A (or of T_A), and $\mathbf{x}$ is said to be an ***eigenvector corresponding to*** $\boldsymbol{\lambda}$.

The requirement that an eigenvector be nonzero is imposed to avoid the unimportant case $A\mathbf{0} = \lambda\mathbf{0}$, which holds for every A and λ.

In general, the image of a vector $\mathbf{x}$ under multiplication by a square matrix A differs from $\mathbf{x}$ in both magnitude and direction. However, in the special case where $\mathbf{x}$ is an eigenvector of A, multiplication by A leaves the direction unchanged. For example, in R^2 or R^3 multiplication by A maps each eigenvector $\mathbf{x}$ of A (if any) along the same line through the origin as $\mathbf{x}$. Depending on the sign and magnitude of the eigenvalue λ

corresponding to $\mathbf{x}$, the operation $A\mathbf{x} = \lambda\mathbf{x}$ compresses or stretches $\mathbf{x}$ by a factor of λ, with a reversal of direction in the case where λ is negative (Figure 5.1.1).



▶ Figure 5.1.1 (*a*) $0 \le \lambda \le 1$ (*b*) $\lambda \ge 1$ (*c*) $-1 \le \lambda \le 0$ (*d*) $\lambda \le -1$

▶ **EXAMPLE 1 Eigenvector of a 2 × 2 Matrix**

$A\mathbf{x} = \lambda \mathbf{x}$

$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

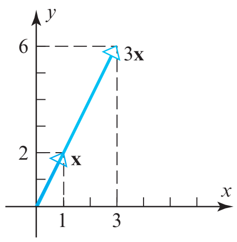
$\lambda\mathbf{x} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 3$, since

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{x}$$



▲ Figure 5.1.2

Geometrically, multiplication by A has stretched the vector $\mathbf{x}$ by a factor of 3 (Figure 5.1.2). ◀

$\boxed{A\mathbf{x} = \lambda \mathbf{x}}$ $\lambda\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x}$

Computing Eigenvalues and Eigenvectors

Our next objective is to obtain a general procedure for finding eigenvalues and eigenvectors of an $n \times n$ matrix A. We will begin with the problem of finding the eigenvalues of A. Note first that the equation $A\mathbf{x} = \lambda\mathbf{x}$ can be rewritten as $A\mathbf{x} = \lambda I\mathbf{x}$, or equivalently, as

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

For λ to be an eigenvalue of A this equation must have a nonzero solution for $\mathbf{x}$. But it follows from parts (*b*) and (*g*) of Theorem 4.10.2 that this is so if and only if the coefficient matrix $\lambda I - A$ has a zero determinant. Thus, we have the following result.

Note that if $(A)_{ij} = a_{ij}$, then formula (1) can be written in expanded form as

$$\begin{vmatrix} \lambda - a_{11} & a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} = 0$$

THEOREM 5.1.1 *If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation*

$$\det(\lambda I - A) = 0 \tag{1}$$

*This is called the **characteristic equation** of A.*

▶ **EXAMPLE 2 Finding Eigenvalues**

In Example 1 we observed that $\lambda = 3$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

but we did not explain how we found it. Use the characteristic equation to find all eigenvalues of this matrix.

$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{bmatrix}$

$\det\begin{bmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{bmatrix} = (\lambda-3)(\lambda+1) - 0 = (\lambda-3)(\lambda+1)$

$\det(\quad) = 0 \Rightarrow (\lambda-3)(\lambda+1) = 0 \Rightarrow (\lambda-3) = 0 \Rightarrow \boxed{\lambda=3}$ or $\lambda+1 = 0 \Rightarrow \boxed{\lambda=-1}$

Solution It follows from Formula (1) that the eigenvalues of A are the solutions of the equation $\det(\lambda I - A) = 0$, which we can write as

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0$$

from which we obtain

$$(\lambda - 3)(\lambda + 1) = 0 \tag{2}$$

This shows that the eigenvalues of A are $\lambda = 3$ and $\lambda = -1$. Thus, in addition to the eigenvalue $\lambda = 3$ noted in Example 1, we have discovered a second eigenvalue $\lambda = -1$. ◀

When the determinant $\det(\lambda I - A)$ in (1) is expanded, the characteristic equation of A takes the form

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0 \tag{3}$$

where the left side of this equation is a polynomial of degree n in which the coefficient of λ^n is 1 (Exercise 37). The polynomial

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n \tag{4}$$

is called the ***characteristic polynomial*** of A. For example, it follows from (2) that the characteristic polynomial of the 2×2 matrix in Example 2 is

$$p(\lambda) = (\lambda - 3)(\lambda + 1) = \lambda^2 - 2\lambda - 3$$

which is a polynomial of degree 2.

Since a polynomial of degree n has at most n distinct roots, it follows from (3) that the characteristic equation of an $n \times n$ matrix A has at most n distinct solutions and consequently the matrix has at most n distinct eigenvalues. Since some of these solutions may be complex numbers, it is possible for a matrix to have complex eigenvalues, even if that matrix itself has real entries. We will discuss this issue in more detail later, but for now we will focus on examples in which the eigenvalues are real numbers.

▶ EXAMPLE 3 Eigenvalues of a 3×3 Matrix

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

The eigenvalues of A must therefore satisfy the cubic equation

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \tag{5}$$

EXAMPLE 3 Eigenvalues of a 3 × 3 Matrix

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$\lambda \begin{vmatrix} \lambda & -1 \\ 17 & \lambda-8 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -4 & \lambda-8 \end{vmatrix} = 0$$

$$\lambda(\lambda(\lambda-8) + 17) - 4 = 0$$

$$\lambda(\lambda^2 - 8\lambda + 17) - 4 = 0$$

$$\boxed{\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0}$$

$$= 0$$

$\pm 2, \pm 4, \pm 1$

$\pm 1, \pm 2, \pm 4$

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 1 \\ \lambda - 4 & \lambda^3 - 8\lambda^2 + 17\lambda - 4 \\ + & -\cancel{\lambda^3} \pm 4\lambda^2 \\ & -4\lambda^2 + 17\lambda - 4 \\ & \pm 4\lambda^2 \mp 16\lambda \\ & \lambda - 4 \\ & +\lambda \mp 4 \\ & 0 \end{array}$$

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

$\lambda - 4 = 0$

$\boxed{\lambda = 4}$

$\lambda^2 - 4\lambda + 1 = 0$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

$\boxed{\lambda = 2 + \sqrt{3}}$ $\boxed{\lambda = 2 - \sqrt{3}}$

To solve this equation, we will begin by searching for integer solutions. This task can be simplified by exploiting the fact that all integer solutions (if there are any) of a polynomial equation with *integer coefficients*

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

must be divisors of the constant term, c_n. Thus, the only possible integer solutions of (5) are the divisors of -4, that is, $\pm1, \pm2, \pm4$. Successively substituting these values in (5) shows that $\lambda = 4$ is an integer solution and hence that $\lambda - 4$ is a factor of the left side of (5). Dividing $\lambda - 4$ into $\lambda^3 - 8\lambda^2 + 17\lambda - 4$ shows that (5) can be rewritten as

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

Thus, the remaining solutions of (5) satisfy the quadratic equation

$$\lambda^2 - 4\lambda + 1 = 0$$

which can be solved by the quadratic formula. Thus, the eigenvalues of A are

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \text{and} \quad \lambda = 2 - \sqrt{3}$$

In applications involving large matrices it is often not feasible to compute the characteristic equation directly, so other methods must be used to find eigenvalues. We will consider such methods in Chapter 9.

▶ EXAMPLE 4 Eigenvalues of an Upper Triangular Matrix

Find the eigenvalues of the upper triangular matrix

$\lambda I - A$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Solution Recalling that the determinant of a triangular matrix is the product of the entries on the main diagonal (Theorem 2.1.2), we obtain

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{bmatrix} \\ &= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) \end{aligned}$$

Thus, the characteristic equation is

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

and the eigenvalues are

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \lambda = a_{33}, \quad \lambda = \boxed{a_{44}}$$

which are precisely the diagonal entries of A. ◀

The following general theorem should be evident from the computations in the preceding example.

THEOREM 5.1.2 *If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.*

EXAMPLE 5 Eigenvalues of a Lower Triangular Matrix

By inspection, the eigenvalues of the lower triangular matrix

Had Theorem 5.1.2 been available earlier, we could have anticipated the result obtained in Example 2.

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

are $\lambda = \frac{1}{2}$, $\lambda = \frac{2}{3}$, and $\lambda = -\frac{1}{4}$. ◀

The following theorem gives some alternative ways of describing eigenvalues.

THEOREM 5.1.3 *If A is an $n \times n$ matrix, the following statements are equivalent.*

(*a*) *λ is an eigenvalue of A.*

(*b*) *λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.*

(*c*) *The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.*

(*d*) *There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.*

Finding Eigenvectors and Bases for Eigenspaces

Now that we know how to find the eigenvalues of a matrix, we will consider the problem of finding the corresponding eigenvectors. By definition, the eigenvectors of A corresponding to an eigenvalue λ are the nonzero vectors that satisfy

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

Notice that $\mathbf{x} = \mathbf{0}$ is in every eigenspace but is not an eigenvector (see Definition 1). In the exercises we will ask you to show that this is the *only* vector that distinct eigenspaces have in common.

Thus, we can find the eigenvectors of A corresponding to λ by finding the nonzero vectors in the solution space of this linear system. This solution space, which is called the ***eigenspace*** of A corresponding to λ, can also be viewed as:

1. the null space of the matrix $\lambda I - A$
2. the kernel of the matrix operator $T_{\lambda I - A}: R^n \to R^n$
3. the set of vectors for which $A\mathbf{x} = \lambda\mathbf{x}$

$\lambda I - A$

EXAMPLE 6 Bases for Eigenspaces

Find bases for the eigenspaces of the matrix

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

Historical Note Methods of linear algebra are used in the emerging field of computerized face recognition. Researchers are working with the idea that every human face in a racial group is a combination of a few dozen primary shapes. For example, by analyzing three-dimensional scans of many faces, researchers at Rockefeller University have produced both an average head shape in the Caucasian group—dubbed the ***meanhead*** (top row left in the figure to the left)—and a set of standardized variations from that shape, called ***eigenheads*** (15 of which are shown in the picture). These are so named because they are eigenvectors of a certain matrix that stores digitized facial information. Face shapes are represented mathematically as linear combinations of the eigenheads.

[*Image: © Dr. Joseph J. Atick, adapted from Scientific American*]

1

Solution The characteristic equation of A is $= \lambda^2 + \lambda - 6$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \left(\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix} \right) \Rightarrow$

$$\begin{vmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{vmatrix} = \lambda(\lambda + 1) - 6 = (\lambda - 2)(\lambda + 3) = 0$$

so the eigenvalues of A are $\lambda = 2$ and $\lambda = -3$. Thus, there are two eigenspaces of A, one for each eigenvalue.

By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an eigenvector of A corresponding to an eigenvalue λ if and only if $(\lambda I - A)\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In the case where $\lambda = 2$ this equation becomes

$$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

whose general solution is

$$x_1 = t, \quad x_2 = t$$

$x_1 - x_2 = 0$

let $x_2 = t$, $t \in R$

$\Rightarrow x_1 = t$

(verify). Since this can be written in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 2$. We leave it for you to follow the pattern of these computations and show that

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

is a basis for the eigenspace corresponding to $\lambda = -3$. ◀

Figure 5.1.3 illustrates the geometric effect of multiplication by the matrix A in Example 6. The eigenspace corresponding to $\lambda = 2$ is the line L_1 through the origin and the point (1, 1), and the eigenspace corresponding to $\lambda = 3$ is the line L_2 through the origin and the point $(-\frac{3}{2}, 1)$. As indicated in the figure, multiplication by A maps each vector in L_1 back into L_1, scaling it by a factor of 2, and it maps each vector in L_2 back into L_2, scaling it by a factor of -3.

▶ EXAMPLE 7 Eigenvectors and Bases for Eigenspaces

Find bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

EXAMPLE 7 Eigenvectors and Bases for Eigenspaces

Find bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$\lambda = 2, 1$

invertiable

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda-2)\begin{vmatrix} \lambda & 2 \\ -1 & \lambda-3 \end{vmatrix} = (\lambda-2)[\lambda(\lambda-3)+2] = 0$$

$$(\lambda-2)(\lambda^2-3\lambda+2) = 0$$

$\lambda - 2 = 0$ or $\lambda^2 - 3\lambda + 2 = 0$

$\boxed{\lambda = 2}$ 1

$(\lambda-1)(\lambda-2) = 0$

$\Rightarrow \boxed{\lambda = 1}$ 2 or $\boxed{\lambda = 2}$ 3

$\boxed{\lambda = 2}$

$$\underbrace{\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0$

let $x_2 = t$, $t \in \mathbb{R}$

let $x_3 = s$, $s \in \mathbb{R}$ $x_2 = t$

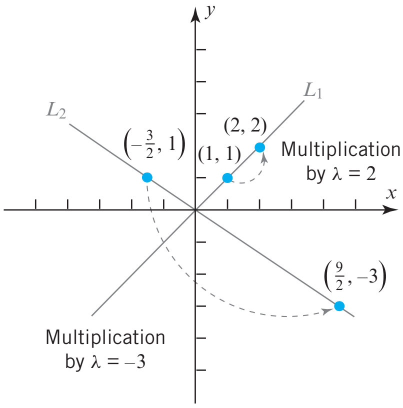
$\Rightarrow x_1 + x_3 = 0 \Rightarrow \boxed{x_1 = -s}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ basis of eigenspace

- $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



▶ Figure 5.1.3

Solution The characteristic equation of A is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$, or in factored form, $(\lambda - 1)(\lambda - 2)^2 = 0$ (verify). Thus, the distinct eigenvalues of A are $\lambda = 1$ and $\lambda = 2$, so there are two eigenspaces of A.

By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is an eigenvector of A corresponding to λ if and only if $\mathbf{x}$ is a nontrivial solution of $(\lambda I - A)\mathbf{x} = \mathbf{0}$, or in matrix form,

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

In the case where $\lambda = 2$, Formula (6) becomes

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system using Gaussian elimination yields (verify)

$$x_1 = -s, \quad x_2 = t, \quad x_3 = s$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent (why?), these vectors form a basis for the eigenspace corresponding to $\lambda = 2$.

If $\lambda = 1$, then (6) becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system yields (verify)

$$x_1 = -2s, \quad x_2 = s, \quad x_3 = s$$

Thus, the eigenvectors corresponding to $\lambda = 1$ are the nonzero vectors of the form

$$\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \text{so that} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 1$. ◀

Eigenvalues and Invertibility

The next theorem establishes a relationship between the eigenvalues and the invertibility of a matrix.

THEOREM 5.1.4 *A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.*

Proof Assume that A is an $n \times n$ matrix and observe first that $\lambda = 0$ is a solution of the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

if and only if the constant term c_n is zero. Thus, it suffices to prove that A is invertible if and only if $c_n \neq 0$. But

$$\det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$$

or, on setting $\lambda = 0$,

$$\det(-A) = c_n \quad \text{or} \quad (-1)^n \det(A) = c_n$$

It follows from the last equation that $\det(A) = 0$ if and only if $c_n = 0$, and this in turn implies that A is invertible if and only if $c_n \neq 0$. ◀

▶ **EXAMPLE 8 Eigenvalues and Invertibility**

The matrix A in Example 7 is invertible since it has eigenvalues $\lambda = 1$ and $\lambda = 2$, neither of which is zero. We leave it for you to check this conclusion by showing that $\det(A) \neq 0$. ◀

More on the Equivalence Theorem

As our final result in this section, we will use Theorem 5.1.4 to add one additional part to Theorem 4.10.2.

THEOREM 5.1.5 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) A *is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row echelon form of* A *is* I_n.

(d) A *is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) *The column vectors of* A *are linearly independent.*

(i) *The row vectors of* A *are linearly independent.*

(j) *The column vectors of* A *span* R^n.

(k) *The row vectors of* A *span* R^n.

(l) *The column vectors of* A *form a basis for* R^n.

(m) *The row vectors of* A *form a basis for* R^n.

(n) A *has rank* n.

(o) A *has nullity* 0.

(p) *The orthogonal complement of the null space of* A *is* R^n.

(q) *The orthogonal complement of the row space of* A *is* $\{\mathbf{0}\}$.

(r) *The kernel of* T_A *is* $\{\mathbf{0}\}$.

(s) *The range of* T_A *is* R^n.

(t) T_A *is one-to-one.*

(u) $\lambda = 0$ *is not an eigenvalue of* A.

Eigenvalues of General Linear Transformations

Thus far, we have only defined eigenvalues and eigenvectors for matrices and linear operators on R^n. The following definition, which parallels Definition 1, extends this concept to general vector spaces.

$Ax = \lambda x$
x: eigenvector of A
λ: eigen value of A

DEFINITION 2 If $T: V \to V$ is a linear operator on a vector space V, then a nonzero vector $\mathbf{x}$ in V is called an ***eigenvector*** of T if $T(\mathbf{x})$ is a scalar multiple of $\mathbf{x}$; that is,

$$T(\mathbf{x}) = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an ***eigenvalue*** of T, and $\mathbf{x}$ is said to be an ***eigenvector corresponding to*** $\boldsymbol{\lambda}$.

$T: V \to V$
$T(x) = \lambda x$

As with matrix operators, we call the kernel of the operator $\lambda I - A$ the ***eigenspace*** of T corresponding to λ. Stated another way, this is the subspace of all vectors in V for which $T(\mathbf{x}) = \lambda\mathbf{x}$.

CALCULUS REQUIRED

In vector spaces of functions eigenvectors are commonly referred to as ***eigenfunctions***.

EXAMPLE 9 Eigenvalue of a Differentiation Operator

If $D: C^\infty \to C^\infty$ is the differentiation operator on the vector space of functions with continuous derivatives of all orders on the interval $(-\infty, \infty)$, and if λ is a constant, then

$$D(e^{\lambda x}) = \lambda e^{\lambda x}$$

so that λ is an eigenvalue of D and $e^{\lambda x}$ is a corresponding eigenvector. ◀

$A(x) = \lambda(x)$
$e^{\lambda x}$

Exercise Set 5.1

In Exercises 1–4, confirm by multiplication that $\mathbf{x}$ is an eigenvector of A, and find the corresponding eigenvalue.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix};\ \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix};\ \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix};\ \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix};\ \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In each part of Exercises 5–6, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

5. (a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

6. (a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

In Exercises 7–12, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

7. $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ **10.** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

In Exercises 13–14, find the characteristic equation of the matrix by inspection.

13. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$ **14.** $\begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

In Exercises 15–16, find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula. [*Suggestion:* Work with the standard matrix for the operator.]

15. $T(x, y) = (x + 4y, 2x + 3y)$

16. $T(x, y, z) = (2x - y - z,\ x - z,\ -x + y + 2z)$

17. (***Calculus required***) Let $D^2\colon C^\infty(-\infty, \infty) \to C^\infty(-\infty, \infty)$ be the operator that maps a function into its second derivative.

(a) Show that D^2 is linear.

(b) Show that if ω is a positive constant, then $\sin\sqrt{\omega}x$ and $\cos\sqrt{\omega}x$ are eigenvectors of D^2, and find their corresponding eigenvalues.

18. (***Calculus required***) Let $D^2\colon C^\infty \to C^\infty$ be the linear operator in Exercise 17. Show that if ω is a positive constant, then $\sinh\sqrt{\omega}x$ and $\cosh\sqrt{\omega}x$ are eigenvectors of D^2, and find their corresponding eigenvalues.

In each part of Exercises 19–20, find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on R^2. Refer to the tables in Section 4.9 and use geometric reasoning to find the answers. No computations are needed.

19. (a) Reflection about the line $y = x$.

(b) Orthogonal projection onto the x-axis.

(c) Rotation about the origin through a positive angle of 90°.

(d) Contraction with factor k $(0 \le k < 1)$.

(e) Shear in the x-direction by a factor k $(k \ne 0)$.

20. (a) Reflection about the y-axis.

(b) Rotation about the origin through a positive angle of 180°.

(c) Dilation with factor k $(k > 1)$.

(d) Expansion in the y-direction with factor k $(k > 1)$.

(e) Shear in the y-direction by a factor k $(k \ne 0)$.

In each part of Exercises 21–22, find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on R^3. Refer to the tables in Section 4.9 and use geometric reasoning to find the answers. No computations are needed.

21. (a) Reflection about the xy-plane.

(b) Orthogonal projection onto the xz-plane.

(c) Counterclockwise rotation about the positive x-axis through an angle of 90°.

(d) Contraction with factor k $(0 \le k < 1)$.

22. (a) Reflection about the xz-plane.

(b) Orthogonal projection onto the yz-plane.

(c) Counterclockwise rotation about the positive y-axis through an angle of 180°.

(d) Dilation with factor k $(k > 1)$.

23. Let A be a 2×2 matrix, and call a line through the origin of R^2 ***invariant*** under A if $A\mathbf{x}$ lies on the line when $\mathbf{x}$ does. Find equations for all lines in R^2, if any, that are invariant under the given matrix.

(a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

24. Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.

(a) $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$

(b) $p(\lambda) = \lambda^4 - \lambda^3 + 7$

[*Hint:* See the proof of Theorem 5.1.4.]

25. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. In each part, answer the question and explain your reasoning.

(a) What is the size of A?

(b) Is A invertible?

(c) How many eigenspaces does A have?

26. The eigenvectors that we have been studying are sometimes called ***right eigenvectors*** to distinguish them from ***left eigenvectors***, which are $n \times 1$ column matrices $\mathbf{x}$ that satisfy the equation $\mathbf{x}^T A = \mu\mathbf{x}^T$ for some scalar μ. For a given matrix A, how are the right eigenvectors and their corresponding eigenvalues related to the left eigenvectors and their corresponding eigenvalues?

27. Find a 3×3 matrix A that has eigenvalues 1, -1, and 0, and for which

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

are their corresponding eigenvectors.

Working with Proofs

28. Prove that the characteristic equation of a 2×2 matrix A can be expressed as $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$, where $\operatorname{tr}(A)$ is the trace of A.

29. Use the result in Exercise 28 to show that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the solutions of the characteristic equation of A are

$$\lambda = \tfrac{1}{2}\left[(a + d) \pm \sqrt{(a - d)^2 + 4bc}\,\right]$$

Use this result to show that A has

(a) two distinct real eigenvalues if $(a - d)^2 + 4bc > 0$.

(b) two repeated real eigenvalues if $(a - d)^2 + 4bc = 0$.

(c) complex conjugate eigenvalues if $(a - d)^2 + 4bc < 0$.

30. Let A be the matrix in Exercise 29. Show that if $b \neq 0$, then

$$\mathbf{x}_1 = \begin{bmatrix} -b \\ a - \lambda_1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -b \\ a - \lambda_2 \end{bmatrix}$$

are eigenvectors of A that correspond, respectively, to the eigenvalues

$$\lambda_1 = \tfrac{1}{2}\left[(a + d) + \sqrt{(a - d)^2 + 4bc}\,\right]$$

and

$$\lambda_2 = \tfrac{1}{2}\left[(a + d) - \sqrt{(a - d)^2 + 4bc}\,\right]$$

31. Use the result of Exercise 28 to prove that if

$$p(\lambda) = \lambda^2 + c_1\lambda + c_2$$

is the characteristic polynomial of a 2×2 matrix, then

$$p(A) = A^2 + c_1 A + c_2 I = 0$$

(Stated informally, A satisfies its characteristic equation. This result is true as well for $n \times n$ matrices.)

32. Prove: If a, b, c, and d are integers such that $a + b = c + d$, then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has integer eigenvalues.

33. Prove: If λ is an eigenvalue of an invertible matrix A and $\mathbf{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1} and $\mathbf{x}$ is a corresponding eigenvector.

34. Prove: If λ is an eigenvalue of A, $\mathbf{x}$ is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$ and $\mathbf{x}$ is a corresponding eigenvector.

35. Prove: If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector, then $s\lambda$ is an eigenvalue of sA for every scalar s and $\mathbf{x}$ is a corresponding eigenvector.

36. Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

and then use Exercises 33 and 34 to find the eigenvalues and bases for the eigenspaces of

(a) A^{-1} (b) $A - 3I$ (c) $A + 2I$

37. Prove that the characteristic polynomial of an $n \times n$ matrix A has degree n and that the coefficient of λ^n in that polynomial is 1.

38. (a) Prove that if A is a square matrix, then A and A^T have the same eigenvalues. [*Hint:* Look at the characteristic equation $\det(\lambda I - A) = 0$.]

(b) Show that A and A^T need not have the same eigenspaces. [*Hint:* Use the result in Exercise 30 to find a 2×2 matrix for which A and A^T have different eigenspaces.]

39. Prove that the intersection of any two distinct eigenspaces of a matrix A is $\{\mathbf{0}\}$.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If A is a square matrix and $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero scalar λ, then $\mathbf{x}$ is an eigenvector of A.

(b) If λ is an eigenvalue of a matrix A, then the linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) If the characteristic polynomial of a matrix A is $p(\lambda) = \lambda^2 + 1$, then A is invertible.

(d) If λ is an eigenvalue of a matrix A, then the eigenspace of A corresponding to λ is the set of eigenvectors of A corresponding to λ.

(e) The eigenvalues of a matrix A are the same as the eigenvalues of the reduced row echelon form of A.

(f) If 0 is an eigenvalue of a matrix A, then the set of columns of A is linearly independent.

Working with Technology

T1. For the given matrix A, find the characteristic polynomial and the eigenvalues, and then use the method of Example 7 to find bases for the eigenspaces.

$$A = \begin{bmatrix} -8 & 33 & 38 & 173 & -30 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & -5 & -25 & 1 \\ 0 & 0 & 1 & 5 & 0 \\ 4 & -16 & -19 & -86 & 15 \end{bmatrix}$$

T2. The Cayley–Hamilton Theorem states that every square matrix satisfies its characteristic equation; that is, if A is an $n \times n$ matrix whose characteristic equation is

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

then $A^n + c_1A^{n-1} + \cdots + c_n = 0$.

(a) Verify the Cayley–Hamilton Theorem for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

(b) Use the result in Exercise 28 to prove the Cayley–Hamilton Theorem for 2×2 matrices.

5.2 Diagonalization

In this section we will be concerned with the problem of finding a basis for R^n that consists of eigenvectors of an $n \times n$ matrix A. Such bases can be used to study geometric properties of A and to simplify various numerical computations. These bases are also of physical significance in a wide variety of applications, some of which will be considered later in this text.

The Matrix Diagonalization Problem

Products of the form $P^{-1}AP$ in which A and P are $n \times n$ matrices and P is invertible will be our main topic of study in this section. There are various ways to think about such products, one of which is to view them as transformations

$$A \rightarrow P^{-1}AP$$

in which the matrix A is mapped into the matrix $P^{-1}AP$. These are called ***similarity transformations***. Such transformations are important because they preserve many properties of the matrix A. For example, if we let $B = P^{-1}AP$, then A and B have the same determinant since

$$\begin{aligned} \det(B) &= \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) = \det(A) \end{aligned}$$