23. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a nonempty set of vectors in an n-dimensional vector space V. Prove that if the vectors in S span V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ span R^n, and conversely.

24. Prove part (*a*) of Theorem 4.5.6.

25. Prove: A subspace of a finite-dimensional vector space is finite-dimensional.

26. State the two parts of Theorem 4.5.2 in contrapositive form.

27. In each part, let S be the standard basis for P_2. Use the results proved in Exercises 22 and 23 to find a basis for the subspace of P_2 spanned by the given vectors.

(a) $-1 + x - 2x^2,\ 3 + 3x + 6x^2,\ 9$

(b) $1 + x,\ x^2,\ 2 + 2x + 3x^2$

(c) $1 + x - 3x^2,\ 2 + 2x - 6x^2,\ 3 + 3x - 9x^2$

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) The zero vector space has dimension zero.

(b) There is a set of 17 linearly independent vectors in R^{17}.

(c) There is a set of 11 vectors that span R^{17}.

(d) Every linearly independent set of five vectors in R^5 is a basis for R^5.

(e) Every set of five vectors that spans R^5 is a basis for R^5.

(f) Every set of vectors that spans R^n contains a basis for R^n.

(g) Every linearly independent set of vectors in R^n is contained in some basis for R^n.

(h) There is a basis for M_{22} consisting of invertible matrices.

(i) If A has size $n \times n$ and $I_n, A, A^2, \ldots, A^{n^2}$ are distinct matrices, then $\{I_n, A, A^2, \ldots, A^{n^2}\}$ is a linearly dependent set.

(j) There are at least two distinct three-dimensional subspaces of P_2.

(k) There are only three distinct two-dimensional subspaces of P_2.

Working with Technology

T1. Devise three different procedures for using your technology utility to determine the dimension of the subspace spanned by a set of vectors in R^n, and then use each of those procedures to determine the dimension of the subspace of R^5 spanned by the vectors

$$\mathbf{v}_1 = (2, 2, -1, 0, 1), \quad \mathbf{v}_2 = (-1, -1, 2, -3, 1),$$
$$\mathbf{v}_3 = (1, 1, -2, 0, -1), \quad \mathbf{v}_4 = (0, 0, 1, 1, 1)$$

T2. Find a basis for the row space of A by starting at the top and successively removing each row that is a linear combination of its predecessors.

$$A = \begin{bmatrix} 3.4 & 2.2 & 1.0 & -1.8 \\ 2.1 & 3.6 & 4.0 & -3.4 \\ 8.9 & 8.0 & 6.0 & 7.0 \\ 7.6 & 9.4 & 9.0 & -8.6 \\ 1.0 & 2.2 & 0.0 & 2.2 \end{bmatrix}$$

4.6 Change of Basis

$S = \{v_1, v_2, \ldots v_n\}$

v relative to S

$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

$c_1, c_2, \ldots c_n$

A basis that is suitable for one problem may not be suitable for another, so it is a common process in the study of vector spaces to change from one basis to another. Because a basis is the vector space generalization of a coordinate system, changing bases is akin to changing coordinate axes in R^2 and R^3. In this section we will study problems related to changing bases.

Coordinate Maps

$(v)_S = (c_1, c_2, \ldots, c_n)$

$[v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V, and if

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

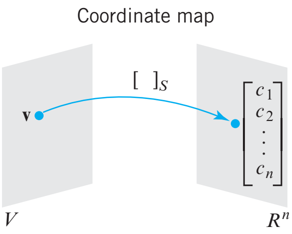
is the coordinate vector of $\mathbf{v}$ relative to S, then, as illustrated in Figure 4.4.6, the mapping

$$\boxed{\mathbf{v} \to (\mathbf{v})_S} \tag{1}$$

creates a connection (a one-to-one correspondence) between vectors in the *general* vector space V and vectors in the *Euclidean* vector space R^n. We call (1) the ***coordinate map relative to S*** from V to R^n. In this section we will find it convenient to express coordinate vectors in the matrix form

$$[\mathbf{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{2}$$

where the square brackets emphasize the matrix notation (Figure 4.6.1).



▲ Figure 4.6.1

Change of Basis

There are many applications in which it is necessary to work with more than one coordinate system. In such cases it becomes important to know how the coordinates of a fixed vector relative to each coordinate system are related. This leads to the following problem.

> **The Change-of-Basis Problem** If $\mathbf{v}$ is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to a basis B', how are the coordinate vectors $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

Remark To solve this problem, it will be convenient to refer to B as the "old basis" and B' as the "new basis." Thus, our objective is to find a relationship between the old and new coordinates of a fixed vector $\mathbf{v}$ in V.

For simplicity, we will solve this problem for two-dimensional spaces. The solution for n-dimensional spaces is similar. Let

new

old $\quad B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$

be the old and new bases, respectively. We will need the coordinate vectors for the new basis vectors relative to the old basis. Suppose they are

$$[\mathbf{u}'_1]_B = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad [\mathbf{u}'_2]_B = \begin{bmatrix} c \\ d \end{bmatrix} \tag{3}$$

That is,

$$\begin{aligned} \mathbf{u}'_1 &= a\mathbf{u}_1 + b\mathbf{u}_2 \quad \text{def} \\ \mathbf{u}'_2 &= c\mathbf{u}_1 + d\mathbf{u}_2 \quad \text{def} \end{aligned} \tag{4}$$

Now let $\mathbf{v}$ be any vector in V, and let

$$[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{5}$$

be the new coordinate vector, so that

$$\mathbf{v} = k_1\mathbf{u}'_1 + k_2\mathbf{u}'_2 \quad \text{def} \tag{6}$$

In order to find the old coordinates of $\mathbf{v}$, we must express $\mathbf{v}$ in terms of the old basis B. To do this, we substitute (4) into (6). This yields

$$\mathbf{v} = k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2)$$

or

$$\mathbf{v} = (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2$$

Thus, the old coordinate vector for $\mathbf{v}$ is

$$[\mathbf{v}]_B = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix}$$

which, by using (5), can be written as

$$[\mathbf{v}]_B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [\mathbf{v}]_{B'}$$

This equation states that the old coordinate vector $[\mathbf{v}]_B$ results when we multiply the new coordinate vector $[\mathbf{v}]_{B'}$ on the left by the matrix

$$P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Since the columns of this matrix are the coordinates of the new basis vectors relative to the old basis [see (3)], we have the following solution of the change-of-basis problem.

from (old)
to (new)

Solution of the Change-of-Basis Problem If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$, then for each vector $\mathbf{v}$ in V, the old coordinate vector $[\mathbf{v}]_B$ is related to the new coordinate vector $[\mathbf{v}]_{B'}$ by the equation

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} \tag{7}$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of P are

$$[\mathbf{u}'_1]_B, \quad [\mathbf{u}'_2]_B, \ldots, \quad [\mathbf{u}'_n]_B \tag{8}$$

Transition Matrices

The matrix P in Equation (7) is called the ***transition matrix*** from B' to B. For emphasis, we will often denote it by $P_{B' \to B}$. It follows from (8) that this matrix can be expressed in terms of its column vectors as

$$P_{B' \to B} = [[\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \cdots \mid [\mathbf{u}'_n]_B] \tag{9}$$

Similarly, the transition matrix from B to B' can be expressed in terms of its column vectors as

$$P_{B \to B'} = [[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \cdots \mid [\mathbf{u}_n]_{B'}] \tag{10}$$

Remark There is a simple way to remember both of these formulas using the terms "old basis" and "new basis" defined earlier in this section: In Formula (9) the old basis is B' and the new basis is B, whereas in Formula (10) the old basis is B and the new basis is B'. Thus, both formulas can be restated as follows:

The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.

▶ EXAMPLE 1 Finding Transition Matrices

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = (1, 0), \quad \mathbf{u}_2 = (0, 1), \quad \mathbf{u}'_1 = (1, 1), \quad \mathbf{u}'_2 = (2, 1)$$

(a) Find the transition matrix $P_{B' \to B}$ from B' to B.

(b) Find the transition matrix $P_{B \to B'}$ from B to B'.

EXAMPLE 1 Finding Transition Matrices

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = (1,0), \quad \mathbf{u}_2 = (0,1), \quad \underset{\text{old}}{\mathbf{u}'_1} = (1,1), \quad \underset{\text{new}}{\mathbf{u}'_2} = (2,1)$$

(a) Find the transition matrix $P_{B' \to B}$ from B' to B.

(b) Find the transition matrix $P_{B \to B'}$ from B to B'.

(a) $P_{B' \to B} = \left[[u'_1]_B \;\; [u'_2]_B \right]$

$[u'_1]_B$: $u'_1 = c_1 u_1 + c_2 u_2$

$(1,1) = c_1(1,0) + c_2(0,1)$

$(1,1) = (c_1, c_2)$

$\boxed{c_1 = 1}$, $\boxed{c_2 = 1}$

$[u'_2]_B$: $u'_2 = c_1 u_1 + c_2 u_2$

$(2,1) = c_1(1,0) + c_2(0,1)$

$(2,1) = (c_1, c_2)$

$\boxed{c_1 = 2}$, $\boxed{c_2 = 1}$

$$P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(b) $P_{B \to B'} = \left[[u_1]_{B'} \;\; [u_2]_{B'} \right]$

$[u_1]_{B'}$: $u_1 = c_1 u'_1 + c_2 u'_2$

$(1,0) = c_1(1,1) + c_2(2,1)$

$(1,0) = (c_1 + 2c_2, c_1 + c_2)$

$$- \begin{cases} c_1 + 2c_2 = 1 \\ c_1 + c_2 = 0 \end{cases}$$

$\boxed{c_2 = 1}$ $\boxed{c_1 = -1}$

$[u_2]_{B'}$ $u_2 = c_1 u'_1 + c_2 u'_2$

$(0,1) = c_1(1,1) + c_2(2,1)$

$(0,1) = (c_1 + 2c_2, c_1 + c_2)$

$$- \begin{cases} c_1 + 2c_2 = 0 \\ c_1 + c_2 = 1 \end{cases}$$

$\boxed{c_2 = -1} \Rightarrow \boxed{c_1 = 2}$

$$P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Solution (a) Here the old basis vectors are $\mathbf{u}_1'$ and $\mathbf{u}_2'$ and the new basis vectors are $\mathbf{u}_1$ and $\mathbf{u}_2$. We want to find the coordinate matrices of the old basis vectors $\mathbf{u}_1'$ and $\mathbf{u}_2'$ relative to the new basis vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. To do this, observe that

$$\begin{aligned}\mathbf{u}_1' &= \mathbf{u}_1 + \mathbf{u}_2\\ \mathbf{u}_2' &= 2\mathbf{u}_1 + \mathbf{u}_2\end{aligned}$$

from which it follows that

$$[\mathbf{u}_1']_B = \begin{bmatrix}1\\1\end{bmatrix} \quad \text{and} \quad [\mathbf{u}_2']_B = \begin{bmatrix}2\\1\end{bmatrix}$$

and hence that

$$P_{B'\to B} = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$$

Solution (b) Here the old basis vectors are $\mathbf{u}_1$ and $\mathbf{u}_2$ and the new basis vectors are $\mathbf{u}_1'$ and $\mathbf{u}_2'$. As in part (a), we want to find the coordinate matrices of the old basis vectors $\mathbf{u}_1'$ and $\mathbf{u}_2'$ relative to the new basis vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. To do this, observe that

$$\begin{aligned}\mathbf{u}_1 &= -\mathbf{u}_1' + \mathbf{u}_2'\\ \mathbf{u}_2 &= \ \ 2\mathbf{u}_1' - \mathbf{u}_2'\end{aligned}$$

from which it follows that

$P_{B\to B'}$

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix}-1\\1\end{bmatrix} \quad \text{and} \quad [\mathbf{u}_2]_{B'} = \begin{bmatrix}2\\-1\end{bmatrix}$$

and hence that

$P_{B'\to B}$

$$P_{B\to B'} = \begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix} \blacktriangleleft$$

Suppose now that B and B' are bases for a finite-dimensional vector space V. Since multiplication by $P_{B'\to B}$ maps coordinate vectors relative to the basis B' into coordinate vectors relative to a basis B, and $P_{B\to B'}$ maps coordinate vectors relative to B into coordinate vectors relative to B', it follows that for every vector $\mathbf{v}$ in V we have

$$[\mathbf{v}]_B = P_{B'\to B}[\mathbf{v}]_{B'} \tag{11}$$

$$[\mathbf{v}]_{B'} = P_{B\to B'}[\mathbf{v}]_B \tag{12}$$

$B=\{u_1, u_2\}$ | $B'=\{u_1', u_2'\}$

$u_1=(1,0)$ | $u_1'=(1,1)$

$u_2=(0,1)$ | $u_2'=(2,1)$

▶ EXAMPLE 2 Computing Coordinate Vectors

(11)

Let B and B' be the bases in Example 1. Use an appropriate formula to find $[\mathbf{v}]_B$ given that

$$[\mathbf{v}]_{B'} = \begin{bmatrix}-3\\5\end{bmatrix}$$ given

Solution To find $[\mathbf{v}]_B$ we need to make the transition from B' to B. It follows from Formula (11) and part (a) of Example 1 that

$$[\mathbf{v}]_B = \boxed{P_{B'\to B}}[\mathbf{v}]_{B'} = \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix} = \begin{bmatrix}7\\2\end{bmatrix} \blacktriangleleft$$

$u_1' = c_1u_1 + c_2u_2$, $u_2' = c_1u_1 + c_2u_2$

Invertibility of Transition Matrices

If B and B' are bases for a finite-dimensional vector space V, then

$$(P_{B'\to B})(P_{B\to B'}) = P_{B\to B}$$

I

because multiplication by the product $(P_{B'\to B})(P_{B\to B'})$ first maps the B-coordinates of a vector into its B'-coordinates, and then maps those B'-coordinates back into the original B-coordinates. Since the net effect of the two operations is to leave each coordinate vector unchanged, we are led to conclude that $P_{B\to B}$ must be the identity matrix, that is,

$$(P_{B'\to B})(P_{B\to B'}) = I \tag{13}$$

(we omit the formal proof). For example, for the transition matrices obtained in Example 1 we have

$$(P_{B'\to B})(P_{B\to B'}) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

It follows from (13) that $P_{B'\to B}$ is invertible and that its inverse is $P_{B\to B'}$. Thus, we have the following theorem.

$P_{B'\to B}$

① invertiable

② P^{-1}: $P_{B\to B'}$

THEOREM 4.6.1 *If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B to B'.*

An Efficient Method for Computing Transition Matrices for R^n

Our next objective is to develop an efficient procedure for computing transition matrices *between bases for* R^n. As illustrated in Example 1, the first step in computing a transition matrix is to express each new basis vector as a linear combination of the old basis vectors. For R^n this involves solving n linear systems of n equations in n unknowns, each of which has the same coefficient matrix (why?). An efficient way to do this is by the method illustrated in Example 2 of Section 1.6, which is as follows:

A Procedure for Computing $P_{B\to B'}$

Step 1. Form the matrix $[B' \mid B]$.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[I \mid P_{B\to B'}]$.

Step 4. Extract the matrix $P_{B\to B'}$ from the right side of the matrix in Step 3.

This procedure is captured in the following diagram.

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}] \tag{14}$$

EXAMPLE 3 Example 1 Revisited

In Example 1 we considered the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = (1, 0), \quad \mathbf{u}_2 = (0, 1), \quad \mathbf{u}'_1 = (1, 1), \quad \mathbf{u}'_2 = (2, 1)$$

(a) Use Formula (14) to find the transition matrix from B' to B.

(b) Use Formula (14) to find the transition matrix from B to B'.

Solution (a) Here B' is the old basis and B is the new basis, so

$$[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

EXAMPLE 3 Example 1 Revisited

In Example 1 we considered the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = (1, 0), \quad \mathbf{u}_2 = (0, 1), \quad \mathbf{u}'_1 = (1, 1), \quad \mathbf{u}'_2 = (2, 1)$$

(a) Use Formula (14) to find the transition matrix from B' to B.

(b) Use Formula (14) to find the transition matrix from B to B'.

(a) $P_{B' \to B}$

① $[\, B \mid B' \,]$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

② $$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$I \qquad P_{B' \to B}$

$$P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(b) $P_{B \to B'}$

$[\, B' \mid B \,]$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R1+R2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-R2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-2R2+R1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$I \qquad ?$

$$\therefore P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \checkmark$$

Since the left side is already the identity matrix, no reduction is needed. We see by inspection that the transition matrix is

$$P_{B'\to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

which agrees with the result in Example 1.

Solution (b) Here B is the old basis and B' is the new basis, so

$$[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

By reducing this matrix, so the left side becomes the identity, we obtain (verify)

$$[I \mid \text{transition from old to new}] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

so the transition matrix is

$$P_{B\to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

which also agrees with the result in Example 1. ◀

Transition to the Standard Basis for R^n

Note that in part (a) of the last example the column vectors of the matrix that made the transition from the basis B' to the standard basis turned out to be the vectors in B' written in column form. This illustrates the following general result.

$[S \mid B'$

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 0 \cdots & 0 & [u_1] & u_2 & u_3 \cdots \\ 0 & 1 & 0 & 0 & & & \\ \vdots & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right]$$

THEOREM 4.6.2 *Let $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be any basis for the vector space R^n and let $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for R^n. If the vectors in these bases are written in column form, then*

$$P_{B'\to S} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n] \tag{15}$$

It follows from this theorem that if

$$A = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n]$$

is any *invertible* $n \times n$ matrix, then A can be viewed as the transition matrix from the basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for R^n to the standard basis for R^n. Thus, for example, the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

which was shown to be invertible in Example 4 of Section 1.5, is the transition matrix from the basis

$$\mathbf{u}_1 = (1, 2, 1), \quad \mathbf{u}_2 = (2, 5, 0), \quad \mathbf{u}_3 = (3, 3, 8)$$

$B' = \{u_1, u_2, u_3\}$

to the basis

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1)$$

$S = \{e_1, e_2, e_3\}$

R^3

$$P_{B'\to S} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Exercise Set 4.6

1. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{u}'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

(a) Find the transition matrix from B' to B.

(b) Find the transition matrix from B to B'.

(c) Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

and use (12) to compute $[\mathbf{w}]_{B'}$.

(d) Check your work by computing $[\mathbf{w}]_{B'}$ directly.

2. Repeat the directions of Exercise 1 with the same vector $\mathbf{w}$ but with

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

3. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{u}'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{u}'_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

(a) Find the transition matrix B to B'.

(b) Compute the coordinate vector $[\mathbf{w}]_B$, where

$$\mathbf{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$$

and use (12) to compute $[\mathbf{w}]_{B'}$.

(c) Check your work by computing $[\mathbf{w}]_{B'}$ directly.

4. Repeat the directions of Exercise 3 with the same vector $\mathbf{w}$, but with

$$\mathbf{u}_1 = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

$$\mathbf{u}'_1 = \begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix}, \quad \mathbf{u}'_2 = \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix}, \quad \mathbf{u}'_3 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

5. Let V be the space spanned by $\mathbf{f}_1 = \sin x$ and $\mathbf{f}_2 = \cos x$.

(a) Show that $\mathbf{g}_1 = 2\sin x + \cos x$ and $\mathbf{g}_2 = 3\cos x$ form a basis for V.

(b) Find the transition matrix from $B' = \{\mathbf{g}_1, \mathbf{g}_2\}$ to $B = \{\mathbf{f}_1, \mathbf{f}_2\}$.

(c) Find the transition matrix from B to B'.

(d) Compute the coordinate vector $[\mathbf{h}]_B$, where $\mathbf{h} = 2\sin x - 5\cos x$, and use (12) to obtain $[\mathbf{h}]_{B'}$.

(e) Check your work by computing $[\mathbf{h}]_{B'}$ directly.

6. Consider the bases $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$ for P_1, where

$$\mathbf{p}_1 = 6 + 3x, \quad \mathbf{p}_2 = 10 + 2x, \quad \mathbf{q}_1 = 2, \quad \mathbf{q}_2 = 3 + 2x$$

(a) Find the transition matrix from B' to B.

(b) Find the transition matrix from B to B'.

(c) Compute the coordinate vector $[\mathbf{p}]_B$, where $\mathbf{p} = -4 + x$, and use (12) to compute $[\mathbf{p}]_{B'}$.

(d) Check your work by computing $[\mathbf{p}]_{B'}$ directly.

7. Let $B_1 = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the bases for R^2 in which $\mathbf{u}_1 = (1, 2)$, $\mathbf{u}_2 = (2, 3)$, $\mathbf{v}_1 = (1, 3)$, and $\mathbf{v}_2 = (1, 4)$.

(a) Use Formula (14) to find the transition matrix $P_{B_2 \to B_1}$.

(b) Use Formula (14) to find the transition matrix $P_{B_1 \to B_2}$.

(c) Confirm that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another.

(d) Let $\mathbf{w} = (0, 1)$. Find $[\mathbf{w}]_{B_1}$ and then use the matrix $P_{B_1 \to B_2}$ to compute $[\mathbf{w}]_{B_2}$ from $[\mathbf{w}]_{B_1}$.

(e) Let $\mathbf{w} = (2, 5)$. Find $[\mathbf{w}]_{B_2}$ and then use the matrix $P_{B_2 \to B_1}$ to compute $[\mathbf{w}]_{B_1}$ from $[\mathbf{w}]_{B_2}$.

8. Let S be the standard basis for R^2, and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis in which $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (-3, 4)$.

(a) Find the transition matrix $P_{B \to S}$ by inspection.

(b) Use Formula (14) to find the transition matrix $P_{S \to B}$.

(c) Confirm that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another.

(d) Let $\mathbf{w} = (5, -3)$. Find $[\mathbf{w}]_B$ and then use Formula (11) to compute $[\mathbf{w}]_S$.

(e) Let $\mathbf{w} = (3, -5)$. Find $[\mathbf{w}]_S$ and then use Formula (12) to compute $[\mathbf{w}]_B$.

9. Let S be the standard basis for R^3, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis in which $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 5, 0)$, and $\mathbf{v}_3 = (3, 3, 8)$.

(a) Find the transition matrix $P_{B \to S}$ by inspection.

(b) Use Formula (14) to find the transition matrix $P_{S \to B}$.

(c) Confirm that $P_{B \to S}$ and $P_{S \to B}$ are inverses of one another.

(d) Let $\mathbf{w} = (5, -3, 1)$. Find $[\mathbf{w}]_B$ and then use Formula (11) to compute $[\mathbf{w}]_S$.

(e) Let $\mathbf{w} = (3, -5, 0)$. Find $[\mathbf{w}]_S$ and then use Formula (12) to compute $[\mathbf{w}]_B$.

10. Let $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for R^2, and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis that results when the vectors in S are reflected about the line $y = x$.

(a) Find the transition matrix $P_{B\to S}$.

(b) Let $P = P_{B\to S}$ and show that $P^T = P_{S\to B}$.

11. Let $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for R^2, and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis that results when the vectors in S are reflected about the line that makes an angle θ with the positive x-axis.

(a) Find the transition matrix $P_{B\to S}$.

(b) Let $P = P_{B\to S}$ and show that $P^T = P_{S\to B}$.

12. If B_1, B_2, and B_3 are bases for R^2, and if

$$P_{B_1\to B_2} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \quad \text{and} \quad P_{B_2\to B_3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}$$

then $P_{B_3\to B_1} =$ ________.

13. If P is the transition matrix from a basis B' to a basis B, and Q is the transition matrix from B to a basis C, what is the transition matrix from B' to C? What is the transition matrix from C to B'?

14. To write the coordinate vector for a vector, it is necessary to specify an order for the vectors in the basis. If P is the transition matrix from a basis B' to a basis B, what is the effect on P if we reverse the order of vectors in B from $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to $\mathbf{v}_n, \ldots, \mathbf{v}_1$? What is the effect on P if we reverse the order of vectors in both B' and B?

15. Consider the matrix

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(a) P is the transition matrix from what basis B to the standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for R^3?

(b) P is the transition matrix from the standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to what basis B for R^3?

16. The matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

is the transition matrix from what basis B to the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ for R^3?

17. Let $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for R^2, and let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis that results when the linear transformation defined by

$$T(x_1, x_2) = (2x_1 + 3x_2, 5x_1 - x_2)$$

is applied to each vector in S. Find the transition matrix $P_{B\to S}$.

18. Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for R^3, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis that results when the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_1 - x_2 + 4x_3, x_2 + 3x_3)$$

is applied to each vector in S. Find the transition matrix $P_{B\to S}$.

19. If $[\mathbf{w}]_B = \mathbf{w}$ holds for all vectors $\mathbf{w}$ in R^n, what can you say about the basis B?

Working with Proofs

20. Let B be a basis for R^n. Prove that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span R^n if and only if the vectors $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ span R^n.

21. Let B be a basis for R^n. Prove that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a linearly independent set in R^n if and only if the vectors $[\mathbf{v}_1]_B, [\mathbf{v}_2]_B, \ldots, [\mathbf{v}_k]_B$ form a linearly independent set in R^n.

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If B_1 and B_2 are bases for a vector space V, then there exists a transition matrix from B_1 to B_2.

(b) Transition matrices are invertible.

(c) If B is a basis for a vector space R^n, then $P_{B\to B}$ is the identity matrix.

(d) If $P_{B_1\to B_2}$ is a diagonal matrix, then each vector in B_2 is a scalar multiple of some vector in B_1.

(e) If each vector in B_2 is a scalar multiple of some vector in B_1, then $P_{B_1\to B_2}$ is a diagonal matrix.

(f) If A is a square matrix, then $A = P_{B_1\to B_2}$ for some bases B_1 and B_2 for R^n.

Working with Technology

T1. Let

$$P = \begin{bmatrix} 5 & 8 & 6 & -13 \\ 3 & -1 & 0 & -9 \\ 0 & 1 & -1 & 0 \\ 2 & 4 & 3 & -5 \end{bmatrix}$$

and

$$\mathbf{v}_1 = (2, 4, 3, -5), \quad \mathbf{v}_2 = (0, 1, -1, 0),$$
$$\mathbf{v}_3 = (3, -1, 0, -9), \quad \mathbf{v}_4 = (5, 8, 6, -13)$$

Find a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ for R^4 for which P is the transition matrix from B to $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

T2. Given that the matrix for a linear transformation $T: R^4 \to R^4$ relative to the standard basis $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ for R^4 is

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

find the matrix for T relative to the basis

$$B' = \{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4\}$$