27. Consider the coordinate vectors

$$[\mathbf{w}]_S = \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix}, \quad [\mathbf{q}]_S = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad [B]_S = \begin{bmatrix} -8 \\ 7 \\ 6 \\ 3 \end{bmatrix}$$

(a) Find $\mathbf{w}$ if S is the basis in Exercise 2.

(b) Find $\mathbf{q}$ if S is the basis in Exercise 3.

(c) Find B if S is the basis in Exercise 5.

28. The basis that we gave for M_{22} in Example 4 consisted of noninvertible matrices. Do you think that there is a basis for M_{22} consisting of invertible matrices? Justify your answer.

Working with Proofs

29. Prove that R^∞ is an infinite-dimensional vector space.

30. Let $T_A\colon R^n \to R^n$ be multiplication by an invertible matrix A, and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for R^n. Prove that $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), \ldots, T_A(\mathbf{u}_n)\}$ is also a basis for R^n.

31. Prove that if V is a subspace of a vector space W and if V is infinite-dimensional, then so is W.

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V.

(b) Every linearly independent subset of a vector space V is a basis for V.

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

(d) The coordinate vector of a vector $\mathbf{x}$ in R^n relative to the standard basis for R^n is $\mathbf{x}$.

(e) Every basis of P_4 contains at least one polynomial of degree 3 or less.

Working with Technology

T1. Let V be the subspace of P_3 spanned by the vectors

$$\mathbf{p}_1 = 1 + 5x - 3x^2 - 11x^3, \quad \mathbf{p}_2 = 7 + 4x - x^2 + 2x^3,$$
$$\mathbf{p}_3 = 5 + x + 9x^2 + 2x^3, \quad \mathbf{p}_4 = 3 - x + 7x^2 + 5x^3$$

(a) Find a basis S for V.

(b) Find the coordinate vector of $\mathbf{p} = 19 + 18x - 13x^2 - 10x^3$ relative to the basis S you obtained in part (a).

T2. Let V be the subspace of $C^\infty(-\infty, \infty)$ spanned by the vectors in the set

$$B = \{1, \cos x, \cos^2 x, \cos^3 x, \cos^4 x, \cos^5 x\}$$

and accept without proof that B is a basis for V. Confirm that the following vectors are in V, and find their coordinate vectors relative to B.

$$\mathbf{f}_0 = 1, \quad \mathbf{f}_1 = \cos x, \quad \mathbf{f}_2 = \cos 2x, \quad \mathbf{f}_3 = \cos 3x,$$
$$\mathbf{f}_4 = \cos 4x, \quad \mathbf{f}_5 = \cos 5x$$

4.5 Dimension

We showed in the previous section that the standard basis for R^n has n vectors and hence that the standard basis for R^3 has three vectors, the standard basis for R^2 has two vectors, and the standard basis for $R^1 (= R)$ has one vector. Since we think of space as three-dimensional, a plane as two-dimensional, and a line as one-dimensional, there seems to be a link between the number of vectors in a basis and the dimension of a vector space. We will develop this idea in this section.

Number of Vectors in a Basis

Our first goal in this section is to establish the following fundamental theorem.

THEOREM 4.5.1 *All bases for a finite-dimensional vector space have the same number of vectors.*

To prove this theorem we will need the following preliminary result, whose proof is deferred to the end of the section.

span lin- inde

S= ✓

THEOREM 4.5.2 *Let V be an n-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis.*

(a) *If a set in V has more than n vectors, then it is linearly dependent.*

(b) *If a set in V has fewer than n vectors, then it does not span V.*

(a) $W=\{w_1, w_2, \ldots w_r\}$ $r>n$

W: depe

(b) $W=\{w_1, w_2, \ldots w_r\}$ $r<n$

W does not span

We can now see rather easily why Theorem 4.5.1 is true; for if

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

is an *arbitrary* basis for V, then the linear independence of S implies that any set in V with more than n vectors is linearly dependent and any set in V with fewer than n vectors does not span V. Thus, unless a set in V has exactly n vectors it cannot be a basis.

We noted in the introduction to this section that for certain familiar vector spaces the intuitive notion of dimension coincides with the number of vectors in a basis. The following definition makes this idea precise.

DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

Engineers often use the term ***degrees of freedom*** as a synonym for dimension.

▶ EXAMPLE 1 Dimensions of Some Familiar Vector Spaces

$\dim(R^n) = n$ **[The standard basis has n vectors.]**

$\dim(P_n) = n + 1$ **[The standard basis has $n + 1$ vectors.]**

$\dim(M_{mn}) = mn$ **[The standard basis has mn vectors.]**

$S=\{1, x, \ldots, x^n\}$

▶ EXAMPLE 2 Dimension of Span(*S*)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ then every vector in span(S) is expressible as a linear combination of the vectors in S. Thus, if the vectors in S are *linearly independent*, they automatically form a basis for span(S), from which we can conclude that

$$\dim[\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}] = r$$

In words, the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

▶ EXAMPLE 3 Dimension of a Solution Space

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \phantom{{}+ 2x_4} + 2x_5 \phantom{{}- 3x_6} &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 + 10x_4 \phantom{{}+ 4x_5} + 15x_6 &= 0 \\
2x_1 + 6x_2 \phantom{{}- 5x_3} + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Solution In Example 6 of Section 1.2 we found the solution of this system to be

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

which can be written in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (\boxed{-3r - 4s - 2t}, \boxed{r}, \boxed{-2s}, \boxed{s}, \boxed{t}, \boxed{0})$$

$= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0)$

$+ t(-2, 0, 0, 0, 1, 0)$

$k_1 v_1 + k_2 v_2 + k_3 v_3 = (0,0,0,0,0,0)$

$-3k_1 - 4k_2 - 2k_3 = 0$

$k_1 = 0$

$-2k_2 = 0$

$k_2 = 0$

$k_3 = 0$

or, alternatively, as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

$S = \{v_1, v_2, v_3\}$

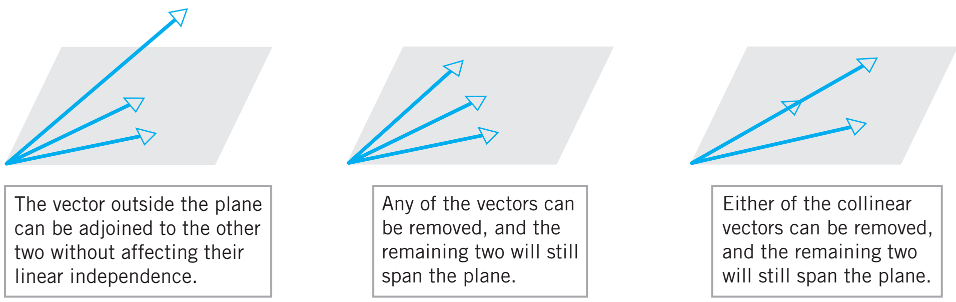
span the solution space. We leave it for you to check that these vectors are linearly independent by showing that none of them is a linear combination of the other two (but see the remark that follows). Thus, the solution space has dimension 3. ◀

Remark It can be shown that for any homogeneous linear system, the method of the last example *always* produces a basis for the solution space of the system. We omit the formal proof.

Some Fundamental Theorems

We will devote the remainder of this section to a series of theorems that reveal the subtle interrelationships among the concepts of linear independence, spanning sets, basis, and dimension. These theorems are not simply exercises in mathematical theory—they are essential to the understanding of vector spaces and the applications that build on them.

We will start with a theorem (proved at the end of this section) that is concerned with the effect on linear independence and spanning if a vector is added to or removed from a nonempty set of vectors. Informally stated, if you start with a linearly independent set S and adjoin to it a vector that is not a linear combination of those already in S, then the enlarged set will still be linearly independent. Also, if you start with a set S of two or more vectors in which one of the vectors is a linear combination of the others, then that vector can be removed from S without affecting span(S) (Figure 4.5.1).



▲ Figure 4.5.1

THEOREM 4.5.3 Plus/Minus Theorem

$S \subseteq V$

$v \in V$ & $v \notin \text{span}(S)$

Let S be a nonempty set of vectors in a vector space V.

(a) *If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.*

(b) *If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,*

$$\boxed{\text{span}(S) = \text{span}(S - \{\mathbf{v}\})}$$

$v \in S$

S span for V

S-{v} span for V

$k_1P_1 + k_2P_2 + k_3P_3 = 0$

$\Rightarrow k_1 = k_2 = k_3 = 0$

EXAMPLE 4 Applying the Plus/Minus Theorem

Show that $\mathbf{p}_1 = 1 - x^2$, $\mathbf{p}_2 = 2 - x^2$, and $\mathbf{p}_3 = x^3$ are linearly independent vectors.

$x^3 \neq k_1P_1 + k_2P_2$

Solution The set $S = \{\mathbf{p}_1, \mathbf{p}_2\}$ is linearly independent since neither vector in S is a scalar multiple of the other. Since the vector $\mathbf{p}_3$ cannot be expressed as a linear combination of the vectors in S (why?), it can be adjoined to S to produce a linearly independent set $S \cup \{\mathbf{p}_3\} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. ◀

In general, to show that a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, one must show that the vectors are linearly independent and span V. However, if we happen to know that V has dimension n (so that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ contains the right number of vectors for a basis), then it suffices to check *either* linear independence *or* spanning—the remaining condition will hold automatically. This is the content of the following theorem.

basis (S)

span　　independ

THEOREM 4.5.4 *Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.*

Proof Assume that S has exactly n vectors and spans V. To prove that S is a basis, we must show that S is a linearly independent set. But if this is not so, then some vector $\mathbf{v}$ in S is a linear combination of the remaining vectors. If we remove this vector from S, then it follows from Theorem 4.5.3(b) that the remaining set of $n - 1$ vectors still spans V. But this is impossible since Theorem 4.5.2(b) states that no set with fewer than n vectors can span an n-dimensional vector space. Thus S is linearly independent.

Assume that S has exactly n vectors and is a linearly independent set. To prove that S is a basis, we must show that S spans V. But if this is not so, then there is some vector $\mathbf{v}$ in V that is not in span(S). If we insert this vector into S, then it follows from Theorem 4.5.3(a) that this set of $n + 1$ vectors is still linearly independent. But this is impossible, since Theorem 4.5.2(a) states that no set with more than n vectors in an n-dimensional vector space can be linearly independent. Thus S spans V. ◀

$|S| = \boxed{2}$　　$S = \{v_1, v_2\}$　　$\dim R^2 = 2$

EXAMPLE 5 Bases by Inspection

(a) Explain why the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2.

(b) Explain why the vectors $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$, and $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for R^3.

$S = \{v_1, v_2, v_3\}$, $|S| = 3$　　$\dim R^3 = 3$

Solution (a) Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space R^2, and hence they form a basis by Theorem 4.5.4.

Solution (b) The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the xz-plane (why?). The vector $\mathbf{v}_3$ is outside of the xz-plane, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. Since R^3 is three-dimensional, Theorem 4.5.4 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the vector space R^3. ◀

$k_1v_1 + k_2v_2 + k_3v_3 = (0, 0, 0)$

$2k_1 + 4k_2 - k_3 = 0$

The next theorem (whose proof is deferred to the end of this section) reveals two important facts about the vectors in a finite-dimensional vector space V:

$\boxed{k_3 = 0}$

$-k_1 + 7k_2 + 4k_3 = 0$

$2k_1 + 4k_2 = 0$

$\rightarrow -k_1 + 7k_2 = 0$

$18k_2 = 0 \Rightarrow \boxed{k_2 = 0} \Rightarrow \boxed{k_1 = 0}$

1. Every spanning set for a subspace is either a basis for that subspace or has a basis as a subset.
2. Every linearly independent set in a subspace is either a basis for that subspace or can be extended to a basis for it.

X basis

span indep X

THEOREM 4.5.5 *Let S be a finite set of vectors in a finite-dimensional vector space V.*

(*a*) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(*b*) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

S: is not span

$S \cup \{v_i\} \Rightarrow$ basis

$S = \{v_1, v_2, \ldots v_r\}$ span

$S - \{v_i\} =$ basis

We conclude this section with a theorem that relates the dimension of a vector space to the dimensions of its subspaces.

THEOREM 4.5.6 *If W is a subspace of a finite-dimensional vector space V, then:*

(*a*) *W is finite-dimensional.*

(*b*) $\dim(W) \le \dim(V)$.

(*c*) *W = V if and only if* $\dim(W) = \dim(V)$.

Proof (a) We will leave the proof of this part as an exercise.

Proof (b) Part (*a*) shows that W is finite-dimensional, so it has a basis

$$S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$$

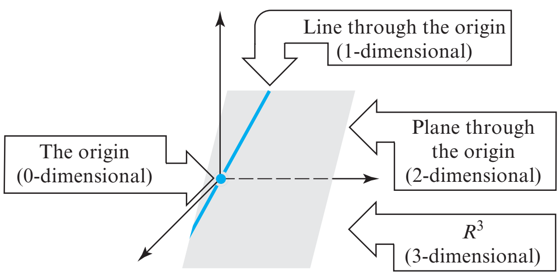
Either S is also a basis for V or it is not. If so, then $\dim(V) = m$, which means that $\dim(V) = \dim(W)$. If not, then because S is a linearly independent set it can be enlarged to a basis for V by part (*b*) of Theorem 4.5.5. But this implies that $\dim(W) < \dim(V)$, so we have shown that $\dim(W) \le \dim(V)$ in all cases.

Proof (c) Assume that $\dim(W) = \dim(V)$ and that

$$S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$$

is a basis for W. If S is not also a basis for V, then being linearly independent S can be extended to a basis for V by part (*b*) of Theorem 4.5.5. But this would mean that $\dim(V) > \dim(W)$, which contradicts our hypothesis. Thus S must also be a basis for V, which means that $W = V$. The converse is obvious. ◀

Figure 4.5.2 illustrates the geometric relationship between the subspaces of R^3 in order of increasing dimension.



▶ Figure 4.5.2

OPTIONAL

We conclude this section with optional proofs of Theorems 4.5.2, 4.5.3, and 4.5.5.

Proof of Theorem 4.5.2(a) Let $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be any set of m vectors in V, where $m > n$. We want to show that S' is linearly dependent. Since $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis, each $\mathbf{w}_i$ can be expressed as a linear combination of the vectors in S, say

$$\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{n1}\mathbf{v}_n \\
\mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{n2}\mathbf{v}_n \\
\vdots \quad & \quad\ \ \vdots \qquad\quad\ \vdots \qquad\qquad\quad \vdots \\
\mathbf{w}_m &= a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + \cdots + a_{nm}\mathbf{v}_n
\end{aligned} \tag{1}$$

To show that S' is linearly dependent, we must find scalars $k_1, k_2, \ldots, k_m$, not all zero, such that

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_m\mathbf{w}_m = \mathbf{0} \tag{2}$$

We leave it for you to verify that the equations in (1) can be rewritten in the partitioned form

$$[\mathbf{w}_1 \mid \mathbf{w}_2 \mid \cdots \mid \mathbf{w}_m] = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n]\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \tag{3}$$

Since $m > n$, the linear system

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4}$$

has more equations than unknowns and hence has a nontrivial solution

$$x_1 = k_1, \quad x_2 = k_2, \ldots, \quad x_m = k_m$$

Creating a column vector from this solution and multiplying both sides of (3) on the right by this vector yields

$$[\mathbf{w}_1 \mid \mathbf{w}_2 \mid \cdots \mid \mathbf{w}_m]\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n]\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}$$

By (4), this simplifies to

$$[\mathbf{w}_1 \mid \mathbf{w}_2 \mid \cdots \mid \mathbf{w}_m]\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which we can rewrite as

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_m\mathbf{w}_m = \mathbf{0}$$

Since the scalar coefficients in this equation are not all zero, we have proved that $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is linearly independent. ◀

The proof of Theorem 4.5.2(b) closely parallels that of Theorem 4.5.2(a) and will be omitted.

Proof of Theorem 4.5.3(a) Assume that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set of vectors in V, and $\mathbf{v}$ is a vector in V that is outside of span(S). To show that $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}\}$ is a linearly independent set, we must show that the only scalars that satisfy

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r + k_{r+1}\mathbf{v} = \mathbf{0} \tag{5}$$

are $k_1 = k_2 = \cdots = k_r = k_{r+1} = 0$. But it must be true that $k_{r+1} = 0$ for otherwise we could solve (5) for $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, contradicting the assumption that $\mathbf{v}$ is outside of span(S). Thus, (5) simplifies to

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0} \tag{6}$$

which, by the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, implies that

$$k_1 = k_2 = \cdots = k_r = 0$$

Proof of Theorem 4.5.3(b) Assume that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of vectors in V, and (to be specific) suppose that $\mathbf{v}_r$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{r-1}$, say

$$\mathbf{v}_r = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{r-1}\mathbf{v}_{r-1} \tag{7}$$

We want to show that if $\mathbf{v}_r$ is removed from S, then the remaining set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{r-1}\}$ still spans S; that is, we must show that every vector $\mathbf{w}$ in span(S) is expressible as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{r-1}\}$. But if $\mathbf{w}$ is in span(S), then $\mathbf{w}$ is expressible in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_{r-1}\mathbf{v}_{r-1} + k_r\mathbf{v}_r$$

or, on substituting (7),

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_{r-1}\mathbf{v}_{r-1} + k_r(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{r-1}\mathbf{v}_{r-1})$$

which expresses $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{r-1}$.

Proof of Theorem 4.5.5(a) If S is a set of vectors that spans V but is not a basis for V, then S is a linearly dependent set. Thus some vector $\mathbf{v}$ in S is expressible as a linear combination of the other vectors in S. By the Plus/Minus Theorem (4.5.3b), we can remove $\mathbf{v}$ from S, and the resulting set S' will still span V. If S' is linearly independent, then S' is a basis for V, and we are done. If S' is linearly dependent, then we can remove some appropriate vector from S' to produce a set S'' that still spans V. We can continue removing vectors in this way until we finally arrive at a set of vectors in S that is linearly independent and spans V. This subset of S is a basis for V.

Proof of Theorem 4.5.5(b) Suppose that $\dim(V) = n$. If S is a linearly independent set that is not already a basis for V, then S fails to span V, so there is some vector $\mathbf{v}$ in V that is not in span(S). By the Plus/Minus Theorem (4.5.3a), we can insert $\mathbf{v}$ into S, and the resulting set S' will still be linearly independent. If S' spans V, then S' is a basis for V, and we are finished. If S' does not span V, then we can insert an appropriate vector into S' to produce a set S'' that is still linearly independent. We can continue inserting vectors in this way until we reach a set with n linearly independent vectors in V. This set will be a basis for V by Theorem 4.5.4. ◀

Exercise Set 4.5

In Exercises 1–6, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

1. $$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

2. $$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

3. $$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 \qquad + 5x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

5. $$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

7. In each part, find a basis for the given subspace of R^3, and state its dimension.
   (a) The plane $3x - 2y + 5z = 0$.
   (b) The plane $x - y = 0$.
   (c) The line $x = 2t$, $y = -t$, $z = 4t$.
   (d) All vectors of the form (a, b, c), where $b = a + c$.

8. In each part, find a basis for the given subspace of R^4, and state its dimension.
   (a) All vectors of the form $(a, b, c, 0)$.
   (b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.
   (c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

9. Find the dimension of each of the following vector spaces.
   (a) The vector space of all diagonal $n \times n$ matrices.
   (b) The vector space of all symmetric $n \times n$ matrices.
   (c) The vector space of all upper triangular $n \times n$ matrices.

10. Find the dimension of the subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

11. (a) Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2.
    (b) Make a conjecture about the dimension of W.
    (c) Confirm your conjecture by finding a basis for W.

12. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.
    (a) $\mathbf{v}_1 = (-1, 2, 3)$, $\mathbf{v}_2 = (1, -2, -2)$
    (b) $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

13. Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.
    $$\mathbf{v}_1 = (1, -4, 2, -3), \quad \mathbf{v}_2 = (-3, 8, -4, 6)$$

14. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

15. The vectors $\mathbf{v}_1 = (1, -2, 3)$ and $\mathbf{v}_2 = (0, 5, -3)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^3.

16. The vectors $\mathbf{v}_1 = (1, 0, 0, 0)$ and $\mathbf{v}_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^4.

17. Find a basis for the subspace of R^3 that is spanned by the vectors
    $$\mathbf{v}_1 = (1, 0, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (2, 0, 1), \quad \mathbf{v}_4 = (0, 0, -1)$$

18. Find a basis for the subspace of R^4 that is spanned by the vectors
    $$\mathbf{v}_1 = (1, 1, 1, 1), \quad \mathbf{v}_2 = (2, 2, 2, 0), \quad \mathbf{v}_3 = (0, 0, 0, 3), \quad \mathbf{v}_4 = (3, 3, 3, 4)$$

19. In each part, let $T_A\colon R^3 \to R^3$ be multiplication by A and find the dimension of the subspace of R^3 consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.
    (a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$
    (c) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

20. In each part, let T_A be multiplication by A and find the dimension of the subspace R^4 consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.
    (a) $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Working with Proofs

21. (a) Prove that for every positive integer n, one can find $n + 1$ linearly independent vectors in $F(-\infty, \infty)$. [*Hint:* Look for polynomials.]
    (b) Use the result in part (a) to prove that $F(-\infty, \infty)$ is infinite-dimensional.
    (c) Prove that $C(-\infty, \infty)$, $C^m(-\infty, \infty)$, and $C^\infty(-\infty, \infty)$ are infinite-dimensional.

22. Let S be a basis for an n-dimensional vector space V. Prove that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ form a linearly independent set of vectors in V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ form a linearly independent set in R^n, and conversely.

$6x + 3y + z = 0$

7. In each part, find a basis for the given subspace of R^3, and state its dimension.

(a) The plane $3x - 2y + 5z = 0$.

(b) The plane $x - y = 0$.

(c) The line $x = 2t$, $y = -t$, $z = 4t$.

(d) All vectors of the form (a, b, c), where $b = a + c$.

$V = \mathbb{R}^3 \quad \dim(\mathbb{R}^3) = 3$

(d) $S = \{(a, b, c),\ b = a + c\}$

$$(a, b, c) = (a, a+c, c) = a(1,1,0) + c(0,1,1)$$

$S = \{v_1, v_2\}$, $v_1 = (1,1,0)$, $v_2 = (0,1,1)$

$\Rightarrow$ S is span $\mathbb{R}^3$ and S in linearly independent because each vector can not be written as multiple the other vector (from theorem)

23. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a nonempty set of vectors in an n-dimensional vector space V. Prove that if the vectors in S span V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ span R^n, and conversely.

24. Prove part (*a*) of Theorem 4.5.6.

25. Prove: A subspace of a finite-dimensional vector space is finite-dimensional.

26. State the two parts of Theorem 4.5.2 in contrapositive form.

27. In each part, let S be the standard basis for P_2. Use the results proved in Exercises 22 and 23 to find a basis for the subspace of P_2 spanned by the given vectors.

(a) $-1 + x - 2x^2,\ 3 + 3x + 6x^2,\ 9$

(b) $1 + x,\ x^2,\ 2 + 2x + 3x^2$

(c) $1 + x - 3x^2,\ 2 + 2x - 6x^2,\ 3 + 3x - 9x^2$

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) The zero vector space has dimension zero.

(b) There is a set of 17 linearly independent vectors in R^{17}.

(c) There is a set of 11 vectors that span R^{17}.

(d) Every linearly independent set of five vectors in R^5 is a basis for R^5.

(e) Every set of five vectors that spans R^5 is a basis for R^5.

(f) Every set of vectors that spans R^n contains a basis for R^n.

(g) Every linearly independent set of vectors in R^n is contained in some basis for R^n.

(h) There is a basis for M_{22} consisting of invertible matrices.

(i) If A has size $n \times n$ and $I_n, A, A^2, \ldots, A^{n^2}$ are distinct matrices, then $\{I_n, A, A^2, \ldots, A^{n^2}\}$ is a linearly dependent set.

(j) There are at least two distinct three-dimensional subspaces of P_2.

(k) There are only three distinct two-dimensional subspaces of P_2.

Working with Technology

T1. Devise three different procedures for using your technology utility to determine the dimension of the subspace spanned by a set of vectors in R^n, and then use each of those procedures to determine the dimension of the subspace of R^5 spanned by the vectors

$$\mathbf{v}_1 = (2, 2, -1, 0, 1), \quad \mathbf{v}_2 = (-1, -1, 2, -3, 1),$$
$$\mathbf{v}_3 = (1, 1, -2, 0, -1), \quad \mathbf{v}_4 = (0, 0, 1, 1, 1)$$

T2. Find a basis for the row space of A by starting at the top and successively removing each row that is a linear combination of its predecessors.

$$A = \begin{bmatrix} 3.4 & 2.2 & 1.0 & -1.8 \\ 2.1 & 3.6 & 4.0 & -3.4 \\ 8.9 & 8.0 & 6.0 & 7.0 \\ 7.6 & 9.4 & 9.0 & -8.6 \\ 1.0 & 2.2 & 0.0 & 2.2 \end{bmatrix}$$

4.6 Change of Basis

A basis that is suitable for one problem may not be suitable for another, so it is a common process in the study of vector spaces to change from one basis to another. Because a basis is the vector space generalization of a coordinate system, changing bases is akin to changing coordinate axes in R^2 and R^3. In this section we will study problems related to changing bases.

Coordinate Maps

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V, and if

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

is the coordinate vector of $\mathbf{v}$ relative to S, then, as illustrated in Figure 4.4.6, the mapping

$$\mathbf{v} \to (\mathbf{v})_S \tag{1}$$

creates a connection (a one-to-one correspondence) between vectors in the *general* vector space V and vectors in the *Euclidean* vector space R^n. We call (1) the ***coordinate map relative to S*** from V to R^n. In this section we will find it convenient to express coordinate