22. Show that for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in a vector space V, the vectors $\mathbf{u}-\mathbf{v}$, $\mathbf{v}-\mathbf{w}$, and $\mathbf{w}-\mathbf{u}$ form a linearly dependent set.

23. (a) In Example 1 we showed that the mutually orthogonal vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ form a linearly independent set of vectors in R^3. Do you think that every set of three nonzero mutually orthogonal vectors in R^3 is linearly independent? Justify your conclusion with a geometric argument.

(b) Justify your conclusion with an algebraic argument. [*Hint:* Use dot products.]

Working with Proofs

24. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors, then so are $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1\}$, $\{\mathbf{v}_2\}$, and $\{\mathbf{v}_3\}$.

25. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

26. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in a vector space V, and $\mathbf{v}_4$ is any vector in V that is not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

27. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set of vectors in a vector space V, and if $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ are any vectors in V that are not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is also linearly dependent.

28. Prove that in P_2 every set with more than three vectors is linearly dependent.

29. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ does not lie in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

30. Use part (*a*) of Theorem 4.3.1 to prove part (*b*).

31. Prove part (*b*) of Theorem 4.3.2.

32. Prove part (*c*) of Theorem 4.3.2.

True-False Exercises

TF. In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) A set containing a single vector is linearly independent.

(b) The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar k.

(c) Every linearly dependent set contains the zero vector.

(d) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ is also linearly independent for every nonzero scalar k.

(e) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.

(f) The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in M_{22}.

(g) The three polynomials $(x-1)(x+2)$, $x(x+2)$, and $x(x-1)$ are linearly independent.

(h) The functions f_1 and f_2 are linearly dependent if there is a real number x such that $k_1 f_1(x) + k_2 f_2(x) = 0$ for some scalars k_1 and k_2.

Working with Technology

T1. Devise three different methods for using your technology utility to determine whether a set of vectors in R^n is linearly independent, and then use each of those methods to determine whether the following vectors are linearly independent.

$$\mathbf{v}_1 = (4, -5, 2, 6), \quad \mathbf{v}_2 = (2, -2, 1, 3),$$
$$\mathbf{v}_3 = (6, -3, 3, 9), \quad \mathbf{v}_4 = (4, -1, 5, 6)$$

T2. Show that $S = \{\cos t, \sin t, \cos 2t, \sin 2t\}$ is a linearly independent set in $C(-\infty, \infty)$ by evaluating the left side of the equation

$$c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t = 0$$

at sufficiently many values of t to obtain a linear system whose only solution is $c_1 = c_2 = c_3 = c_4 = 0$.

4.4 Coordinates and Basis

- span
- Independent.

V $\quad$ $S=\{v_1, v_2, \ldots v_r\}$

We usually think of a line as being one-dimensional, a plane as two-dimensional, and the space around us as three-dimensional. It is the primary goal of this section and the next to make this intuitive notion of dimension precise. In this section we will discuss coordinate systems in general vector spaces and lay the groundwork for a precise definition of dimension in the next section.

(a, b, c)

Coordinate Systems in Linear Algebra

In analytic geometry one uses *rectangular* coordinate systems to create a one-to-one correspondence between points in 2-space and ordered pairs of real numbers and between points in 3-space and ordered triples of real numbers (Figure 4.4.1). Although rectangular coordinate systems are common, they are not essential. For example, Figure 4.4.2 shows coordinate systems in 2-space and 3-space in which the coordinate axes are not mutually perpendicular.

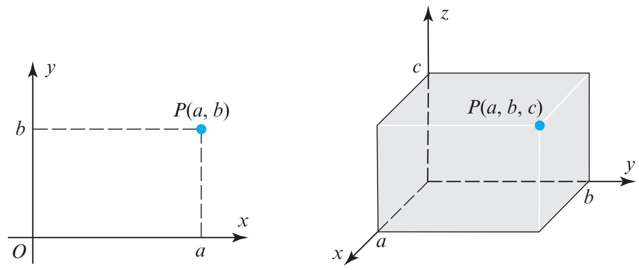


Figure 4.4.1 Coordinates of P in a rectangular coordinate system in 2-space. Coordinates of P in a rectangular coordinate system in 3-space.

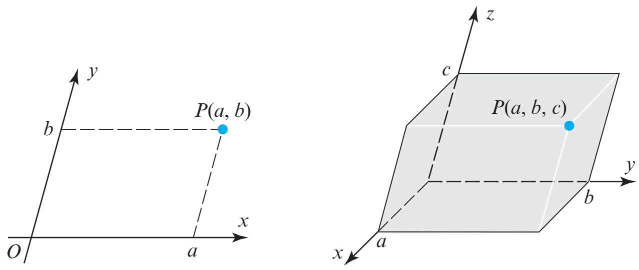


Figure 4.4.2 Coordinates of P in a nonrectangular coordinate system in 2-space. Coordinates of P in a nonrectangular coordinate system in 3-space.

In linear algebra coordinate systems are commonly specified using vectors rather than coordinate axes. For example, in Figure 4.4.3 we have re-created the coordinate systems in Figure 4.4.2 by using unit vectors to identify the positive directions and then attaching coordinates to a point P using the scalar coefficients in the equations

$$\overrightarrow{OP} = a\mathbf{u}_1 + b\mathbf{u}_2 \quad \text{and} \quad \overrightarrow{OP} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$$

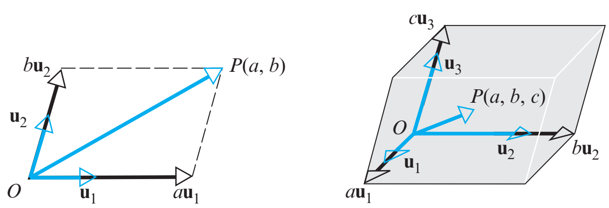
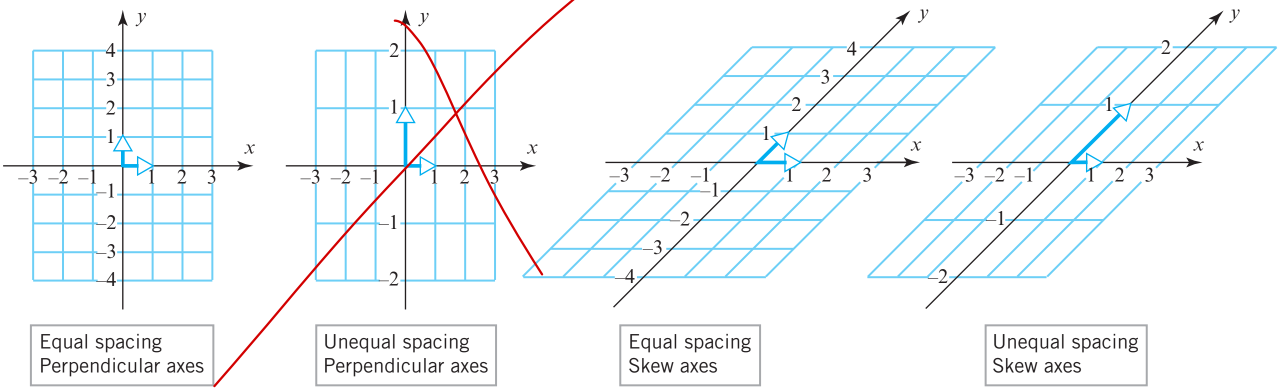


Figure 4.4.3

Units of measurement are essential ingredients of any coordinate system. In geometry problems one tries to use the same unit of measurement on all axes to avoid distorting the shapes of figures. This is less important in applications where coordinates represent physical quantities with diverse units (for example, time in seconds on one axis and temperature in degrees Celsius on another axis). To allow for this level of generality, we will relax the requirement that *unit* vectors be used to identify the positive directions and require only that those vectors be linearly independent. We will refer to these as the "basis vectors" for the coordinate system. In summary, it is the directions of the basis vectors that establish the positive directions, and it is the lengths of the basis vectors that establish the spacing between the integer points on the axes (Figure 4.4.4).



▲ Figure 4.4.4

Basis for a Vector Space

Our next goal is to extend the concepts of "basis vectors" and "coordinate systems" to general vector spaces, and for that purpose we will need some definitions. Vector spaces fall into two categories: A vector space V is said to be ***finite-dimensional*** if there is a finite set of vectors in V that spans V and is said to be ***infinite-dimensional*** if no such set exists.

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a ***basis*** for V if:

(a) S spans V.

(b) S is linearly independent.

If you think of a basis as describing a coordinate system for a finite-dimensional vector space V, then part (a) of this definition guarantees that there are enough basis vectors to provide coordinates for all vectors in V, and part (b) guarantees that there is no interrelationship between the basis vectors. Here are some examples.

▶ EXAMPLE 1 The Standard Basis for R^n

Recall from Example 11 of Section 4.2 that the standard unit vectors

$S=\{e_1, e_2, \dots e_n\}$

span ind-

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

span R^n and from Example 1 of Section 4.3 that they are linearly independent. Thus, they form a basis for R^n that we call the ***standard basis for R^n***. In particular,

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

is the standard basis for R^3.

▶ EXAMPLE 2 The Standard Basis for P_n

Show that $S = \{1, x, x^2, \dots, x^n\}$ is a basis for the vector space P_n of polynomials of degree n or less.

Solution We must show that the polynomials in S are linearly independent and span P_n. Let us denote these polynomials by

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \dots, \quad \mathbf{p}_n = x^n$$

We showed in Example 13 of Section 4.2 that these vectors span P_n and in Example 4 of Section 4.3 that they are linearly independent. Thus, they form a basis for P_n that we call the ***standard basis for P_n***.

▶ EXAMPLE 3 Another Basis for R^3

Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for R^3.

Solution We must show that these vectors are linearly independent and span R^3. To prove linear independence we must show that the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{1}$$

has only the trivial solution; and to prove that the vectors span R^3 we must show that every vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b} \tag{2}$$

By equating corresponding components on the two sides, these two equations can be expressed as the linear systems

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 \qquad + 4c_3 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad + 4c_3 &= b_3 \end{aligned} \tag{3}$$

(verify). Thus, we have reduced the problem to showing that in (3) the homogeneous system has only the trivial solution and that the nonhomogeneous system is consistent for all values of b_1, b_2, and b_3. But the two systems have the same coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

From Examples 1 and 3 you can see that a vector space can have more than one basis.

so it follows from parts (b), (e), and (g) of Theorem 2.3.8 that we can prove both results at the same time by showing that $\det(A) \neq 0$. We leave it for you to confirm that $\det(A) = -1$, which proves that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for R^3.

▶ EXAMPLE 4 The Standard Basis for M_{mn}

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

Solution We must show that the matrices are linearly independent and span M_{22}. To prove linear independence we must show that the equation

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \mathbf{0} \tag{4}$$

has only the trivial solution, where $\mathbf{0}$ is the 2×2 zero matrix; and to prove that the matrices span M_{22} we must show that every 2×2 matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

can be expressed as

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = B \tag{5}$$

The matrix forms of Equations (4) and (5) are

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

linear inda

and

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Span

EXAMPLE 3 Another Basis for R^3

Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for R^3.

$S = \{v_1, v_2, v_3\}$, $V = R^3$

① span

$$v = (a,b,c) = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$(a,b,c) = k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4)$$

$$(a,b,c) = (k_1 + 2k_2 + 3k_3 ,\ 2k_1 + 9k_2 + 3k_3 ,\ k_1 + 4k_3)$$

$$\left.\begin{aligned} k_1 + 2k_2 + 3k_3 &= a \\ 2k_1 + 9k_2 + 3k_3 &= b \\ k_1 + 4k_3 &= c \end{aligned}\right\} \ (*)$$

$$A = \begin{bmatrix} 1^{+} & 2 & 3 \\ 2^{-} & 9 & 3 \\ \boxed{1^{+} \quad 0^{-} \quad 4^{+}} \end{bmatrix}, \quad \det(A) = 1 \begin{vmatrix} 2 & 3 \\ 9 & 3 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix}$$

$$= 1(6 - 27) + 4(9 - 4)$$

$$= -21 + 20 = -1 \neq 0$$

(by Theorem

the system $*$ has a solution $\Longrightarrow$ S span R^3

② linearly independent.

$$(0,0,0) = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$(0,0,0) = k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4)$$

$$(0,0,0) = (k_1 + 2k_2 + 3k_3 ,\ 2k_1 + 9k_2 + 3k_3 ,\ k_1 + 4k_3)$$

$$k_1 + 2k_2 + 3k_3 = 0$$
$$2k_1 + 9k_2 + 3k_3 = 0$$
$$k_1 + 4k_3 = 0$$

since $\det A \neq 0 \Rightarrow Ax = 0$ has only a trivial solution

$$\Rightarrow k_1 = k_2 = k_3 = 0$$

S is linearly independent

from ① & ② S forms a basis for R^3

which can be rewritten as

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since the first equation has only the trivial solution

$$c_1 = c_2 = c_3 = c_4 = 0$$

the matrices are linearly independent, and since the second equation has the solution

$$c_1 = a, \quad c_2 = b, \quad c_3 = c, \quad c_4 = d$$

the matrices span M_{22}. This proves that the matrices M_1, M_2, M_3, M_4 form a basis for M_{22}. More generally, the mn different matrices whose entries are zero except for a single entry of 1 form a basis for M_{mn} called the ***standard basis for*** $\boldsymbol{M_{mn}}$. ◀

The simplest of all vector spaces is the zero vector space $V = \{\mathbf{0}\}$. This space is finite-dimensional because it is spanned by the vector $\mathbf{0}$. However, it has no basis in the sense of Definition 1 because $\{\mathbf{0}\}$ is not a linearly independent set (why?). However, we will find it useful to define the empty set Ø to be a basis for this vector space.

$V = P_\infty$

degree : $0, \ldots, \infty$

▶ EXAMPLE 5 An Infinite-Dimensional Vector Space

Show that the vector space of P_∞ of all polynomials with real coefficients is infinite-dimensional by showing that it has no finite spanning set.

$S = \{P_1, P_2, \ldots, P_r\}$

degree $\le n$

Solution If there were a finite spanning set, say $S = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_r\}$, then the degrees of the polynomials in S would have a maximum value, say n; and this in turn would imply that any linear combination of the polynomials in S would have degree at most n. Thus, there would be no way to express the polynomial x^{n+1} as a linear combination of the polynomials in S, contradicting the fact that the vectors in S span P_∞.

$x^{n+1} \neq k_1 P_1 + k_2 P_2 + \cdots + k_r P_r$

▶ EXAMPLE 6 Some Finite- and Infinite-Dimensional Spaces

In Examples 1, 2, and 4 we found bases for R^n, P_n, and M_{mn}, so these vector spaces are finite-dimensional. We showed in Example 5 that the vector space P_∞ is not spanned by finitely many vectors and hence is infinite-dimensional. Some other examples of infinite-dimensional vector spaces are R^∞, $F(-\infty, \infty)$, $C(-\infty, \infty)$, $C^m(-\infty, \infty)$, and $C^\infty(-\infty, \infty)$. ◀

Coordinates Relative to a Basis

Earlier in this section we drew an informal analogy between basis vectors and coordinate systems. Our next goal is to make this informal idea precise by defining the notion of a coordinate system in a general vector space. The following theorem will be our first step in that direction.

THEOREM 4.4.1 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.

Proof Since S spans V, it follows from the definition of a spanning set that every vector in V is expressible as a linear combination of the vectors in S. To see that there is only *one* way to express a vector as a linear combination of the vectors in S, suppose that some vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

and also as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Subtracting the second equation from the first gives

$$\mathbf{0} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \cdots + (c_n - k_n)\mathbf{v}_n$$

Since the right side of this equation is a linear combination of vectors in S, the linear independence of S implies that

$$c_1 - k_1 = 0, \quad c_2 - k_2 = 0, \ldots, \quad c_n - k_n = 0$$

that is,

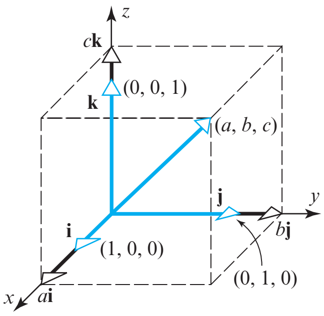
$$c_1 = k_1, \quad c_2 = k_2, \ldots, \quad c_n = k_n$$

Thus, the two expressions for $\mathbf{v}$ are the same. ◀

We now have all of the ingredients required to define the notion of "coordinates" in a general vector space V. For motivation, observe that in R^3, for example, the coordinates (a, b, c) of a vector $\mathbf{v}$ are precisely the coefficients in the formula

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

that expresses $\mathbf{v}$ as a linear combination of the standard basis vectors for R^3 (see Figure 4.4.5). The following definition generalizes this idea.



▲ Figure 4.4.5

DEFINITION 2 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

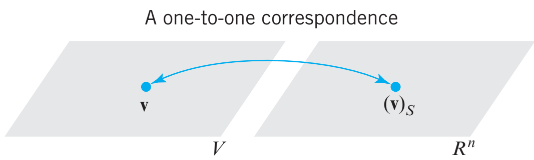
is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the ***coordinates* of v** relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the ***coordinate vector of* v *relative to S***; it is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n) \tag{6}$$

Sometimes it will be desirable to write a coordinate vector as a column matrix or row matrix, in which case we will denote it with square brackets as $[\mathbf{v}]_S$. We will refer to this as the ***matrix form*** of the coordinate vector and (6) as the ***comma-delimited form***.

Remark It is standard to regard two *sets* to be the same if they have the same members, even if those members are written in a different order. In particular, in a basis for a vector space V, which is a *set* of linearly independent vectors that span V, the order in which those vectors are listed does not generally matter. However, the order in which they are listed is critical for coordinate vectors, since changing the order of the basis vectors changes the coordinate vectors [for example, in R^2 the coordinate pair (1, 2) is not the same as the coordinate pair (2, 1)]. To deal with this complication, many authors define an ***ordered basis*** to be one in which the listing order of the basis vectors remains fixed. In all discussions involving coordinate vectors we will assume that the underlying basis is ordered, even though we may not say so explicitly.

Observe that $(\mathbf{v})_S$ is a vector in R^n, so that once an ordered basis S is given for a vector space V, Theorem 4.4.1 establishes a one-to-one correspondence between vectors in V and vectors in R^n (Figure 4.4.6).



▶ Figure 4.4.6

▶ EXAMPLE 7 Coordinates Relative to the Standard Basis for R^n

In the special case where $V = R^n$ and S is the *standard basis*, the coordinate vector $(\mathbf{v})_S$ and the vector $\mathbf{v}$ are the same; that is,

$$\mathbf{v} = (\mathbf{v})_S$$

$\boxed{(a,b,c)} = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

For example, in R^3 the representation of a vector $\mathbf{v} = (a, b, c)$ as a linear combination of the vectors in the standard basis $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is

$i = (1,0,0)$ $k = (0,0,1)$
$j = (0,1,0)$

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$c_1 = a$
$c_2 = b$
$c_3 = c$
$(v)_S = (a,b,c)$

so the coordinate vector relative to this basis is $(\mathbf{v})_S = (a, b, c)$, which is the same as the vector $\mathbf{v}$.

▶ EXAMPLE 8 Coordinate Vectors Relative to Standard Bases

(a) Find the coordinate vector for the polynomial

$$\mathbf{p}(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

relative to the standard basis for the vector space P_n. $S = \{1, x, x^2, \ldots, x^n\}$

8 (a)

$P(x) = k_0 1 + k_1 x + k_2 x^2 + \cdots + k_n x^n$

(b) Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22}. $(B)_S = (a,b,c,d)$

$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = k_0 + k_1 x + k_2 x^2 + \cdots + k_n x^n$

$k_0 = c_0$
$k_1 = c_1$
$k_2 = c_2$
$\vdots$
$k_n = c_n$

Solution (a) The given formula for $\mathbf{p}(x)$ expresses this polynomial as a linear combination of the standard basis vectors $S = \{1, x, x^2, \ldots, x^n\}$. Thus, the coordinate vector for $\mathbf{p}$ relative to S is

$$(\mathbf{p})_S = (c_0, c_1, c_2, \ldots, c_n)$$

Solution (b) We showed in Example 4 that the representation of a vector

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as a linear combination of the standard basis vectors is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so the coordinate vector of B relative to S is

$$(B)_S = (a, b, c, d)$$

▶ EXAMPLE 9 Coordinates in R^3

(a) We showed in Example 3 that the vectors

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

form a basis for R^3. Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Find the vector $\mathbf{v}$ in R^3 whose coordinate vector relative to S is $(\mathbf{v})_S = (-1, 3, 2)$.

Solution (a) To find $(\mathbf{v})_S$ we must first express $\mathbf{v}$ as a linear combination of the vectors in S; that is, we must find values of c_1, c_2, and c_3 such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

EXAMPLE 9 Coordinates in R^3

$S=\{v_1, v_2, v_3\}$ basis

(a) We showed in Example 3 that the vectors

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (2, 9, 0), \quad \mathbf{v}_3 = (3, 3, 4)$$

form a basis for R^3. Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

a b c

(b) Find the vector $\mathbf{v}$ in R^3 whose coordinate vector relative to S is $(\mathbf{v})_S = (-1, 3, 2)$.

$c_1\ c_2\ c_3$

ⓐ $\boxed{(5,-1,9)} = c_1v_1 + c_2v_2 + c_3v_3$

$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$

$(5,-1,9) = (c_1+2c_2+3c_3,\ 2c_1+9c_2+3c_3,\ c_1+4c_3)$

$c_1+2c_2+3c_3 = 5$

$2c_1+9c_2+3c_3 = -1$

$c_1+4c_3 = 9$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$

$\det A = -1 \neq 0$

$Ax = b$

$x = A^{-1}b$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{array}\right] \xrightarrow[-1R1+R3]{-2R1+R2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 5 & -3 & -11 \\ 0 & -2 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & -\frac{3}{5} & -\frac{11}{5} \\ 0 & -2 & 1 & 4 \end{array}\right] \xrightarrow[2R2+R3]{-2R2+R1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{21}{5} & \frac{47}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{11}{5} \\ 0 & 0 & -\frac{1}{5} & -\frac{2}{5} \end{array}\right]$$

$$\xrightarrow{-5R3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{21}{5} & \frac{47}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{11}{5} \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-\frac{21}{5}R3+R1]{\frac{3}{5}R3+R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$c = 1$

$c_2 = -1$

$c_3 = 2$

$(v)_S = (1,-1,2)$

ⓑ $\boxed{v} = c_1v_1 + c_2v_2 + c_3v_3$

$v = -1 \cdot v_1 + 3v_2 + 2 \cdot v_3 = -1(1,2,1) + 3(2,9,0) + 2(3,3,4)$

$v = (-1,-2,-1) + (6,27,0) + (6,6,8)$

$\boxed{v = (11,31,7)}$

or, in terms of components,

$$(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$$

Equating corresponding components gives

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 \qquad\quad + 4c_3 &= 9 \end{aligned}$$

Solving this system we obtain $c_1 = 1$, $c_2 = -1$, $c_3 = 2$ (verify). Therefore,

$$(\mathbf{v})_S = (1, -1, 2)$$

Solution (b) Using the definition of $(\mathbf{v})_S$, we obtain

$$\begin{aligned} \mathbf{v} &= (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 \\ &= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) = (11, 31, 7) \end{aligned}$$

Exercise Set 4.4

1. Use the method of Example 3 to show that the following set of vectors forms a basis for R^2.
$$\{(2, 1), (3, 0)\}$$

2. Use the method of Example 3 to show that the following set of vectors forms a basis for R^3.
$$\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$$

3. Show that the following polynomials form a basis for P_2.
$$x^2 + 1, \quad x^2 - 1, \quad 2x - 1$$

4. Show that the following polynomials form a basis for P_3.
$$1 + x, \quad 1 - x, \quad 1 - x^2, \quad 1 - x^3$$

5. Show that the following matrices form a basis for M_{22}.
$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

6. Show that the following matrices form a basis for M_{22}.
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

7. In each part, show that the set of vectors is not a basis for R^3.
   (a) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$
   (b) $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

8. Show that the following vectors do not form a basis for P_2.
$$1 - 3x + 2x^2, \quad 1 + x + 4x^2, \quad 1 - 7x$$

9. Show that the following matrices do not form a basis for M_{22}.
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

10. Let V be the space spanned by $\mathbf{v}_1 = \cos^2 x$, $\mathbf{v}_2 = \sin^2 x$, $\mathbf{v}_3 = \cos 2x$.
   (a) Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for V.
   (b) Find a basis for V.

11. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.
   (a) $\mathbf{u}_1 = (2, -4)$, $\mathbf{u}_2 = (3, 8)$; $\mathbf{w} = (1, 1)$
   (b) $\mathbf{u}_1 = (1, 1)$, $\mathbf{u}_2 = (0, 2)$; $\mathbf{w} = (a, b)$

12. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.
   (a) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (1, 0)$
   (b) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (0, 1)$

13. Find the coordinate vector of $\mathbf{v}$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3.
   (a) $\mathbf{v} = (2, -1, 3)$; $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (2, 2, 0)$, $\mathbf{v}_3 = (3, 3, 3)$
   (b) $\mathbf{v} = (5, -12, 3)$; $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

14. Find the coordinate vector of $\mathbf{p}$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2.
   (a) $\mathbf{p} = 4 - 3x + x^2$; $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, $\mathbf{p}_3 = x^2$
   (b) $\mathbf{p} = 2 - x + x^2$; $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 1 + x^2$, $\mathbf{p}_3 = x + x^2$

In Exercises 15–16, first show that the set $S = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

15. $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$,
$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

16. $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
$A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$

In Exercises 17–18, first show that the set $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for P_2, then express $\mathbf{p}$ as a linear combination of the vectors in S, and then find the coordinate vector of $\mathbf{p}$ relative to S.

17. $\mathbf{p}_1 = 1 + x + x^2$, $\mathbf{p}_2 = x + x^2$, $\mathbf{p}_3 = x^2$;
$\mathbf{p} = 7 - x + 2x^2$

18. $\mathbf{p}_1 = 1 + 2x + x^2$, $\mathbf{p}_2 = 2 + 9x$, $\mathbf{p}_3 = 3 + 3x + 4x^2$;
$\mathbf{p} = 2 + 17x - 3x^2$

19. In words, explain why the sets of vectors in parts (a) to (d) are *not* bases for the indicated vector spaces.

(a) $\mathbf{u}_1 = (1, 2)$, $\mathbf{u}_2 = (0, 3)$, $\mathbf{u}_3 = (1, 5)$ for R^2

(b) $\mathbf{u}_1 = (-1, 3, 2)$, $\mathbf{u}_2 = (6, 1, 1)$ for R^3

(c) $\mathbf{p}_1 = 1 + x + x^2$, $\mathbf{p}_2 = x$ for P_2

(d) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix}$,
$D = \begin{bmatrix} 5 & 0 \\ 4 & 2 \end{bmatrix}$ for M_{22}

20. In any vector space a set that contains the zero vector must be linearly dependent. Explain why this is so.

21. In each part, let $T_A\colon R^3 \to R^3$ be multiplication by A, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for R^3. Determine whether the set $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_3)\}$ is linearly independent in R^2.

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

22. In each part, let $T_A\colon R^3 \to R^3$ be multiplication by A, and let $\mathbf{u} = (1, -2, -1)$. Find the coordinate vector of $T_A(\mathbf{u})$ relative to the basis $S = \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ for R^3.

(a) $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

23. The accompanying figure shows a rectangular xy-coordinate system determined by the unit basis vectors $\mathbf{i}$ and $\mathbf{j}$ and an $x'y'$-coordinate system determined by unit basis vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Find the $x'y'$-coordinates of the points whose xy-coordinates are given.

(a) $(\sqrt{3}, 1)$ (b) $(1, 0)$ (c) $(0, 1)$ (d) (a, b)

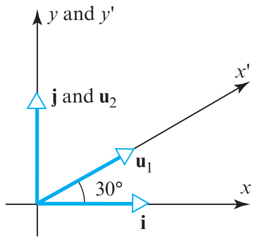


Figure Ex-23

24. The accompanying figure shows a rectangular xy-coordinate system and an $x'y'$-coordinate system with skewed axes. Assuming that 1-unit scales are used on all the axes, find the $x'y'$-coordinates of the points whose xy-coordinates are given.

(a) $(1, 1)$ (b) $(1, 0)$ (c) $(0, 1)$ (d) (a, b)

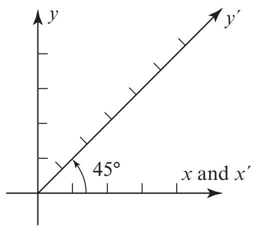


Figure Ex-24

25. The first four ***Hermite polynomials*** [named for the French mathematician Charles Hermite (1822–1901)] are

$$1, \quad 2t, \quad -2 + 4t^2, \quad -12t + 8t^3$$

These polynomials have a wide variety of applications in physics and engineering.

(a) Show that the first four Hermite polynomials form a basis for P_3.

(b) Let B be the basis in part (a). Find the coordinate vector of the polynomial

$$\mathbf{p}(t) = -1 - 4t + 8t^2 + 8t^3$$

relative to B.

26. The first four ***Laguerre polynomials*** [named for the French mathematician Edmond Laguerre (1834–1886)] are

$$1, \quad 1 - t, \quad 2 - 4t + t^2, \quad 6 - 18t + 9t^2 - t^3$$

(a) Show that the first four Laguerre polynomials form a basis for P_3.

(b) Let B be the basis in part (a). Find the coordinate vector of the polynomial

$$\mathbf{p}(t) = -10t + 9t^2 - t^3$$

relative to B.

27. Consider the coordinate vectors

$$[\mathbf{w}]_S = \begin{bmatrix} 6 \\ -1 \\ 4 \end{bmatrix}, \quad [\mathbf{q}]_S = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad [B]_S = \begin{bmatrix} -8 \\ 7 \\ 6 \\ 3 \end{bmatrix}$$

(a) Find $\mathbf{w}$ if S is the basis in Exercise 2.

(b) Find $\mathbf{q}$ if S is the basis in Exercise 3.

(c) Find B if S is the basis in Exercise 5.

28. The basis that we gave for M_{22} in Example 4 consisted of noninvertible matrices. Do you think that there is a basis for M_{22} consisting of invertible matrices? Justify your answer.

Working with Proofs

29. Prove that R^∞ is an infinite-dimensional vector space.

30. Let $T_A: R^n \to R^n$ be multiplication by an invertible matrix A, and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for R^n. Prove that $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), \ldots, T_A(\mathbf{u}_n)\}$ is also a basis for R^n.

31. Prove that if V is a subspace of a vector space W and if V is infinite-dimensional, then so is W.

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V.

(b) Every linearly independent subset of a vector space V is a basis for V.

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

(d) The coordinate vector of a vector $\mathbf{x}$ in R^n relative to the standard basis for R^n is $\mathbf{x}$.

(e) Every basis of P_4 contains at least one polynomial of degree 3 or less.

Working with Technology

T1. Let V be the subspace of P_3 spanned by the vectors

$$\mathbf{p}_1 = 1 + 5x - 3x^2 - 11x^3, \quad \mathbf{p}_2 = 7 + 4x - x^2 + 2x^3,$$
$$\mathbf{p}_3 = 5 + x + 9x^2 + 2x^3, \quad \mathbf{p}_4 = 3 - x + 7x^2 + 5x^3$$

(a) Find a basis S for V.

(b) Find the coordinate vector of $\mathbf{p} = 19 + 18x - 13x^2 - 10x^3$ relative to the basis S you obtained in part (a).

T2. Let V be the subspace of $C^\infty(-\infty, \infty)$ spanned by the vectors in the set

$$B = \{1, \cos x, \cos^2 x, \cos^3 x, \cos^4 x, \cos^5 x\}$$

and accept without proof that B is a basis for V. Confirm that the following vectors are in V, and find their coordinate vectors relative to B.

$$\mathbf{f}_0 = 1, \quad \mathbf{f}_1 = \cos x, \quad \mathbf{f}_2 = \cos 2x, \quad \mathbf{f}_3 = \cos 3x,$$
$$\mathbf{f}_4 = \cos 4x, \quad \mathbf{f}_5 = \cos 5x$$

4.5 Dimension

We showed in the previous section that the standard basis for R^n has n vectors and hence that the standard basis for R^3 has three vectors, the standard basis for R^2 has two vectors, and the standard basis for $R^1 (= R)$ has one vector. Since we think of space as three-dimensional, a plane as two-dimensional, and a line as one-dimensional, there seems to be a link between the number of vectors in a basis and the dimension of a vector space. We will develop this idea in this section.

Number of Vectors in a Basis

Our first goal in this section is to establish the following fundamental theorem.

THEOREM 4.5.1 *All bases for a finite-dimensional vector space have the same number of vectors.*

To prove this theorem we will need the following preliminary result, whose proof is deferred to the end of the section.