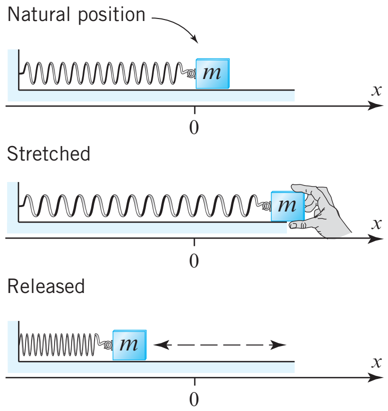
where ω is a fixed constant that depends on the mass of the block and the stiffness of the spring and c_1 and c_2 are arbitrary. Show that this set of functions forms a subspace of $C^\infty(-\infty, \infty)$.



▲ Figure Ex-23

Working with Proofs

24. Prove Theorem 4.2.6.

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Every subspace of a vector space is itself a vector space.

(b) Every vector space is a subspace of itself.

(c) Every subset of a vector space V that contains the zero vector in V is a subspace of V.

(d) The kernel of a matrix transformation $T_A: R^n \to R^m$ is a subspace of R^m.

(e) The solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns is a subspace of R^n.

(f) The span of any finite set of vectors in a vector space is closed under addition and scalar multiplication.

(g) The intersection of any two subspaces of a vector space V is a subspace of V.

(h) The union of any two subspaces of a vector space V is a subspace of V.

(i) Two subsets of a vector space V that span the same subspace of V must be equal.

(j) The set of upper triangular $n \times n$ matrices is a subspace of the vector space of all $n \times n$ matrices.

(k) The polynomials $x - 1$, $(x - 1)^2$, and $(x - 1)^3$ span P_3.

Working with Technology

T1. Recall from Theorem 1.3.1 that a product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of the matrix A in which the coefficients are the entries of $\mathbf{x}$. Use matrix multiplication to compute

$$\mathbf{v} = 6(8, -2, 1, -4) + 17(-3, 9, 11, 6) - 9(13, -1, 2, 4)$$

T2. Use the idea in Exercise T1 and matrix multiplication to determine whether the polynomial

$$\mathbf{p} = 1 + x + x^2 + x^3$$

is in the span of

$$\mathbf{p}_1 = 8 - 2x + x^2 - 4x^3, \quad \mathbf{p}_2 = -3 + 9x + 11x^2 + 6x^3,$$
$$\mathbf{p}_3 = 13 - x + 2x^2 + 4x^3$$

T3. For the vectors that follow, determine whether

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

$$\mathbf{v}_1 = (-1, 2, 0, 1, 3), \quad \mathbf{v}_2 = (7, 4, 6, -3, 1),$$
$$\mathbf{v}_3 = (-5, 3, 1, 2, 4)$$
$$\mathbf{w}_1 = (-6, 5, 1, 3, 7), \quad \mathbf{w}_2 = (6, 6, 6, -2, 4),$$
$$\mathbf{w}_3 = (2, 7, 7, -1, 5)$$

4.3 Linear Independence

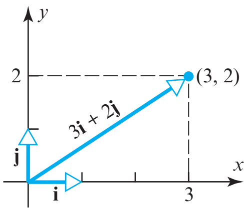
In this section we will consider the question of whether the vectors in a given set are interrelated in the sense that one or more of them can be expressed as a linear combination of the others. This is important to know in applications because the existence of such relationships often signals that some kind of complication is likely to occur.

Linear Independence and Dependence

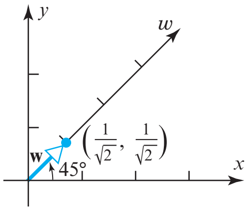
In a rectangular xy-coordinate system every vector in the plane can be expressed in exactly one way as a linear combination of the standard unit vectors. For example, the only way to express the vector (3, 2) as a linear combination of $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ is

$$(3, 2) = 3(1, 0) + 2(0, 1) = 3\mathbf{i} + 2\mathbf{j} \tag{1}$$

R^2



▲ Figure 4.3.1



▲ Figure 4.3.2

(Figure 4.3.1). Suppose, however, that we were to introduce a third coordinate axis that makes an angle of 45° with the x-axis. Call it the w-axis. As illustrated in Figure 4.3.2, the unit vector along the w-axis is

x y

$$\mathbf{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

i j w
(1,0) (0,1) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Whereas Formula (1) shows the only way to express the vector (3, 2) as a linear combination of **i** and **j**, there are infinitely many ways to express this vector as a linear combination of **i**, **j**, and **w**. Three possibilities are

k_1 k_2 k_3

$$(3, 2) = 3(1, 0) + 2(0, 1) + 0\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 3\mathbf{i} + 2\mathbf{j} + 0\mathbf{w}$$

$$(3, 2) = 2(1, 0) + (0, 1) + \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 3\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{w}$$

$$(3, 2) = 4(1, 0) + 3(0, 1) - \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 4\mathbf{i} + 3\mathbf{j} - \sqrt{2}\mathbf{w}$$

In short, by introducing a superfluous axis we created the complication of having multiple ways of assigning coordinates to points in the plane. What makes the vector **w** superfluous is the fact that it can be expressed as a linear combination of the vectors **i** and **j**, namely,

$$\mathbf{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

This leads to the following definition.

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a ***linearly independent set*** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be ***linearly dependent***.

In the case where the set S in Definition 1 has only one vector, we will agree that S is linearly independent if and only if that vector is nonzero.

In general, the most efficient way to determine whether a set is linearly independent or not is to use the following theorem whose proof is given at the end of this section.

THEOREM 4.3.1 *A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

▶ EXAMPLE 1 Linear Independence of the Standard Unit Vectors in R^n

The most basic linearly independent set in R^n is the set of standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

To illustrate this in R^3, consider the standard unit vectors

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

$K_1(1,0,0) + K_2(0,1,0) + K_3(0,0,1) = (0,0,0)$

$(K_1, K_2, K_3) = (0,0,0)$

$K_1 = 0, \ K_2 = 0, \ K_3 = 0$

To prove linear independence we must show that the only coefficients satisfying the vector equation

$$k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, k_3 = 0$. But this becomes evident by writing this equation in its component form

$$(k_1, k_2, k_3) = (0, 0, 0)$$

You should have no trouble adapting this argument to establish the linear independence of the standard unit vectors in R^n.

EXAMPLE 2 Linear Independence in R^3

Determine whether the vectors $S = \{v_1, v_2, v_3\}$

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1) \tag{2}$$

are linearly independent or linearly dependent in R^3.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0} \tag{3}$$

can be satisfied with coefficients that are not all zero. To see whether this is so, let us rewrite (3) in the component form

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned} \tag{4}$$

$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$

Thus, our problem reduces to determining whether this system has nontrivial solutions. There are various ways to do this; one possibility is to simply solve the system, which yields

det(A) = 0

A is not invertible ⇒ many solutions

$$k_1 = -\tfrac{1}{2}t, \quad k_2 = -\tfrac{1}{2}t, \quad k_3 = t$$

(we omit the details). This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent. A second method for establishing the linear dependence is to take advantage of the fact that the coefficient matrix

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

is square and compute its determinant. We leave it for you to show that $\det(A) = 0$ from which it follows that (4) has nontrivial solutions by parts (*b*) and (*g*) of Theorem 2.3.8.

Because we have established that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in (2) are linearly dependent, we know that at least one of them is a linear combination of the others. We leave it for you to confirm, for example, that

$$\mathbf{v}_3 = \tfrac{1}{2}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2$$

EXAMPLE 3 Linear Independence in R^4

Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

in R^4 are linearly dependent or linearly independent.

Solution The linear independence or linear dependence of these vectors is determined by whether there exist nontrivial solutions of the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

or, equivalently, of

$$k_1(1, 2, 2, -1) + k_2(4, 9, 9, -4) + k_3(5, 8, 9, -5) = (0, 0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

$$\begin{array}{cc} & 4 \quad 3 \\ A = & \begin{bmatrix} 1 & 4 & 5 \\ 2 & 9 & 8 \\ 2 & 9 & 9 \\ -1 & -4 & -5 \end{bmatrix} \end{array} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We leave it for you to show that this system has only the trivial solution

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

from which you can conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. ◀

EXAMPLE 4 An Important Linearly Independent Set in P_n

Show that the polynomials

$$S = \{1, x, x^2, \ldots, x^n\}$$

form a linearly independent set in P_n.

Solution For convenience, let us denote the polynomials as

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \ldots, \quad \mathbf{p}_n = x^n$$

We must show that the only coefficients satisfying the vector equation

$$a_0\mathbf{p}_0 + a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + \cdots + a_n\mathbf{p}_n = \mathbf{0} \to 0 + 0\cdot x + 0\cdot x^2 + \cdots + 0x^n \tag{5}$$

are

$$a_0 = a_1 = a_2 = \cdots = a_n = 0$$

$a_0 \cdot 1 = 0$

$a_0 = 0$

$a_1 = 0$

But (5) is equivalent to the statement that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0 \tag{6}$$

for all x in $(-\infty, \infty)$, so we must show that this is true if and only if each coefficient in (6) is zero. To see that this is so, recall from algebra that a nonzero polynomial of degree n has at most n distinct roots. That being the case, each coefficient in (6) must be zero, for otherwise the left side of the equation would be a nonzero polynomial with infinitely many roots. Thus, (5) has only the trivial solution. ◀

The following example shows that the problem of determining whether a given set of vectors in P_n is linearly independent or linearly dependent can be reduced to determining whether a certain set of vectors in R^n is linearly dependent or independent.

▶ EXAMPLE 5 Linear Independence of Polynomials

P_2

Determine whether the polynomials

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in P_2.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0} \tag{7}$$

can be satisfied with coefficients that are not all zero. To see whether this is so, let us rewrite (7) in its polynomial form

$0 + 0\cdot x + 0\cdot x^2$

$$k_1(1 - x) + k_2(5 + 3x - 2x^2) + k_3(1 + 3x - x^2) = 0 \tag{8}$$

or, equivalently, as

$$(k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

Since this equation must be satisfied by all x in $(-\infty, \infty)$, each coefficient must be zero (as explained in the previous example). Thus, the linear dependence or independence of the given polynomials hinges on whether the following linear system has a nontrivial solution:

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned} \tag{9}$$

$A = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix}$

We leave it for you to show that this linear system has nontrivial solutions either by solving it directly or by showing that the coefficient matrix has determinant zero. Thus, the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent. ◀

$\det(A) = 0$
$\Rightarrow$ infinitely many

> In Example 5, what relationship do you see between the coefficients of the given polynomials and the column vectors of the coefficient matrix of system (9)?

Sets with One or Two Vectors

The following useful theorem is concerned with the linear independence and linear dependence of sets with one or two vectors and sets that contain the zero vector.

THEOREM 4.3.2

(*a*) *A finite set that contains* $\mathbf{0}$ *is linearly dependent.*

(*b*) *A set with exactly one vector is linearly independent if and only if that vector is not* $\mathbf{0}$.

(*c*) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

We will prove part (*a*) and leave the rest as exercises.

Proof (a) For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{0}\}$ is linearly dependent since the equation

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r + 1(\mathbf{0}) = \mathbf{0}$$

expresses $\mathbf{0}$ as a linear combination of the vectors in S with coefficients that are not all zero. ◀

▶ EXAMPLE 6 Linear Independence of Two Functions

The functions $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \sin x$ are linearly independent vectors in $F(-\infty, \infty)$ since neither function is a scalar multiple of the other. On the other hand, the two functions $\mathbf{g}_1 = \sin 2x$ and $\mathbf{g}_2 = \sin x \cos x$ are linearly dependent because the trigonometric identity $\sin 2x = 2 \sin x \cos x$ reveals that $\mathbf{g}_1$ and $\mathbf{g}_2$ are scalar multiples of each other. ◀

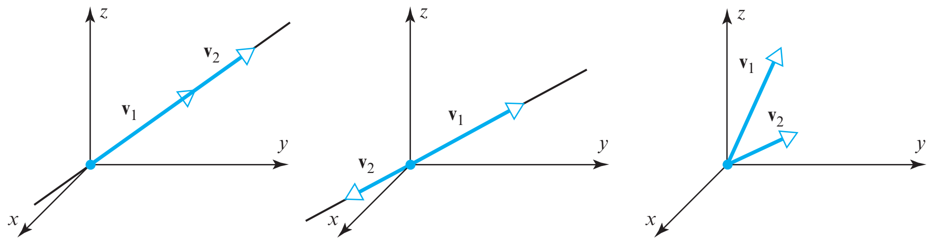
$S = \{x, \sin x\}$ $\quad x \neq k_1 \sin x$
$\quad \sin x \neq k_2 x$

$S = \{\sin 2x, \sin x \cos x\}$, $\sin 2x = 2 \sin x \cos x$

A Geometric Interpretation of Linear Independence

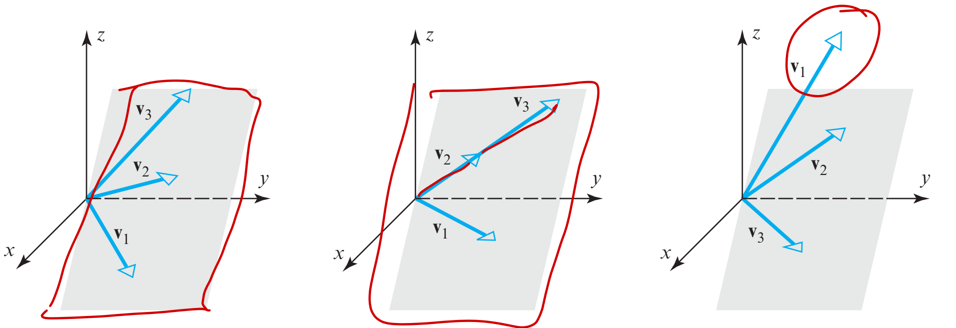
Linear independence has the following useful geometric interpretations in R^2 and R^3:

- Two vectors in R^2 or R^3 are linearly independent if and only if they do not lie on the same line when they have their initial points at the origin. Otherwise one would be a scalar multiple of the other (Figure 4.3.3).



▶ Figure 4.3.3 (a) Linearly dependent (b) Linearly dependent (c) Linearly independent

- Three vectors in R^3 are linearly independent if and only if they do not lie in the same plane when they have their initial points at the origin. Otherwise at least one would be a linear combination of the other two (Figure 4.3.4).



▶ Figure 4.3.4 (a) Linearly dependent (b) Linearly dependent (c) Linearly independent

At the beginning of this section we observed that a third coordinate axis in R^2 is superfluous by showing that a unit vector along such an axis would have to be expressible as a linear combination of unit vectors along the positive x- and y-axis. That result is a consequence of the next theorem, which shows that there can be at most n vectors in any linearly independent set R^n.

$p \to q$

$\equiv \neg q \to \neg p$

$r > n \to$ dep

indep $\to r \leq n$

THEOREM 4.3.3 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.*

r vectors

Proof Suppose that

$$\begin{aligned}
\mathbf{v}_1 &= (v_{11}, v_{12}, \ldots, v_{1n}) \\
\mathbf{v}_2 &= (v_{21}, v_{22}, \ldots, v_{2n}) \\
&\ \ \vdots \qquad\qquad \vdots \\
\mathbf{v}_r &= (v_{r1}, v_{r2}, \ldots, v_{rn})
\end{aligned}$$

and consider the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

It follows from Theorem 4.3.3 that a set in R^2 with more than two vectors is linearly dependent and a set in R^3 with more than three vectors is linearly dependent.

If we express both sides of this equation in terms of components and then equate the corresponding components, we obtain the system

$$\begin{aligned} v_{11}k_1 + v_{21}k_2 + \cdots + v_{r1}k_r &= 0 \\ v_{12}k_1 + v_{22}k_2 + \cdots + v_{r2}k_r &= 0 \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad &\quad \vdots \\ v_{1n}k_1 + v_{2n}k_2 + \cdots + v_{rn}k_r &= 0 \end{aligned}$$

This is a homogeneous system of n equations in the r unknowns $k_1, \dots, k_r$. Since $r > n$, it follows from Theorem 1.2.2 that the system has nontrivial solutions. Therefore, $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a linearly dependent set. ◀

CALCULUS REQUIRED

Linear Independence of Functions

Sometimes linear dependence of functions can be deduced from known identities. For example, the functions

$$\mathbf{f}_1 = \sin^2 x, \quad \mathbf{f}_2 = \cos^2 x, \quad \text{and} \quad \mathbf{f}_3 = 5$$

form a linearly dependent set in $F(-\infty, \infty)$, since the equation

$$\begin{aligned} 5\mathbf{f}_1 + 5\mathbf{f}_2 - \mathbf{f}_3 &= 5\sin^2 x + 5\cos^2 x - 5 \\ &= 5(\sin^2 x + \cos^2 x) - 5 = \mathbf{0} \end{aligned}$$

expresses $\mathbf{0}$ as a linear combination of $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ with coefficients that are not all zero.

However, it is relatively rare that linear independence or dependence of functions can be ascertained by algebraic or trigonometric methods. To make matters worse, there is no general method for doing that either. That said, there does exist a theorem that can be useful for that purpose in certain cases. The following definition is needed for that theorem.

DEFINITION 2 If $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \dots, \mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \dots, f_n$.

Józef Hoëné de Wroński (1778–1853)

Historical Note The Polish-French mathematician Józef Hoëné de Wroński was born Józef Hoëné and adopted the name Wroński after he married. Wroński's life was fraught with controversy and conflict, which some say was due to psychopathic tendencies and his exaggeration of the importance of his own work. Although Wroński's work was dismissed as rubbish for many years, and much of it was indeed erroneous, some of his ideas contained hidden brilliance and have survived. Among other things, Wroński designed a caterpillar vehicle to compete with trains (though it was never manufactured) and did research on the famous problem of determining the longitude of a ship at sea. His final years were spent in poverty.

[Image: © TopFoto/The Image Works]

Suppose for the moment that $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \dots, \mathbf{f}_n = f_n(x)$ are *linearly dependent* vectors in $C^{(n-1)}(-\infty, \infty)$. This implies that the vector equation

$$k_1\mathbf{f}_1 + k_2\mathbf{f}_2 + \cdots + k_n\mathbf{f}_n = \mathbf{0}$$

is satisfied by values of the coefficients $k_1, k_2, \dots, k_n$ that are not all zero, and for these coefficients the equation

$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0$$

is satisfied for all x in $(-\infty, \infty)$. Using this equation together with those that result by differentiating it $n-1$ times we obtain the linear system

$$\begin{aligned}
k_1 f_1(x) &+ k_2 f_2(x) &+ \cdots + k_n f_n(x) &= 0\\
k_1 f_1'(x) &+ k_2 f_2'(x) &+ \cdots + k_n f_n'(x) &= 0\\
\vdots \quad & \quad \vdots & \vdots \quad & \quad \vdots\\
k_1 f_1^{(n-1)}(x) &+ k_2 f_2^{(n-1)}(x) &+ \cdots + k_n f_n^{(n-1)}(x) &= 0
\end{aligned}$$

Thus, the linear dependence of $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ implies that the linear system

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

has a nontrivial solution for every x in the interval $(-\infty, \infty)$, and this in turn implies that the determinant of the coefficient matrix of (10) is zero for every such x. Since this determinant is the Wronskian of $f_1, f_2, \dots, f_n$, we have established the following result.

WARNING The converse of Theorem 4.3.4 is false. If the Wronskian of $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ is identically zero on $(-\infty, \infty)$, then no conclusion can be reached about the linear independence of $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$—this set of vectors may be linearly independent or linearly dependent.

THEOREM 4.3.4 *If the functions* $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ *have* $n-1$ *continuous derivatives on the interval* $(-\infty, \infty)$, *and if the Wronskian of these functions is not identically zero on* $(-\infty, \infty)$, *then these functions form a linearly independent set of vectors in* $C^{(n-1)}(-\infty, \infty)$.

$\neq 0$

In Example 6 we showed that x and $\sin x$ are linearly independent functions by observing that neither is a scalar multiple of the other. The following example illustrates how to obtain the same result using the Wronskian (though it is a more complicated procedure in this particular case).

EXAMPLE 7 Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \sin x$ are linearly independent vectors in $C^{\infty}(-\infty, \infty)$.

Solution The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \quad \neq 0$$

This function is not identically zero on the interval $(-\infty, \infty)$ since, for example,

$$W\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Thus, the functions are linearly independent.

$2 \cdot e^{2x} \cdot 2$
$= 4e^{2x}$

EXAMPLE 8 Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = 1$, $\mathbf{f}_2 = e^x$, and $\mathbf{f}_3 = e^{2x}$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

Solution The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 2e^{3x} \neq 0$$

functions

first der

second der

$1 \;\; \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix}$

$4e^{3x} - 2e^{3x} = 2e^{3x}$

This function is obviously not identically zero on $(-\infty, \infty)$, so $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ form a linearly independent set. ◀

OPTIONAL

We will close this section by proving Theorem 4.3.1.

Proof of Theorem 4.3.1 We will prove this theorem in the case where the set S has two or more vectors, and leave the case where S has only one vector as an exercise. Assume first that S is linearly independent. We will show that if the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0} \tag{11}$$

can be satisfied with coefficients that are not all zero, then at least one of the vectors in S must be expressible as a linear combination of the others, thereby contradicting the assumption of linear independence. To be specific, suppose that $k_1 \neq 0$. Then we can rewrite (11) as

$$\mathbf{v}_1 = \left(-\frac{k_2}{k_1}\right)\mathbf{v}_2 + \cdots + \left(-\frac{k_r}{k_1}\right)\mathbf{v}_r$$

which expresses $\mathbf{v}_1$ as a linear combination of the other vectors in S.

Conversely, we must show that if the only coefficients satisfying (11) are

$$k_1 = 0, \quad k_2 = 0, \ldots, \quad k_r = 0$$

then the vectors in S must be linearly independent. But if this were true of the coefficients and the vectors were not linearly independent, then at least one of them would be expressible as a linear combination of the others, say

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r$$

which we can rewrite as

$$\mathbf{v}_1 + (-c_2)\mathbf{v}_2 + \cdots + (-c_r)\mathbf{v}_r = \mathbf{0}$$

But this contradicts our assumption that (11) can only be satisfied by coefficients that are all zero. Thus, the vectors in S must be linearly independent. ◀

Exercise Set 4.3

1. Explain why the following form linearly dependent sets of vectors. (Solve this problem by inspection.)

   (a) $\mathbf{u}_1 = (-1, 2, 4)$ and $\mathbf{u}_2 = (5, -10, -20)$ in R^3

   (b) $\mathbf{u}_1 = (3, -1)$, $\mathbf{u}_2 = (4, 5)$, $\mathbf{u}_3 = (-4, 7)$ in R^2

   (c) $\mathbf{p}_1 = 3 - 2x + x^2$ and $\mathbf{p}_2 = 6 - 4x + 2x^2$ in P_2

   (d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

2. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^3.

   (a) $(-3, 0, 4)$, $(5, -1, 2)$, $(1, 1, 3)$

   (b) $(-2, 0, 1)$, $(3, 2, 5)$, $(6, -1, 1)$, $(7, 0, -2)$

3. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^4.

   (a) $(3, 8, 7, -3)$, $(1, 5, 3, -1)$, $(2, -1, 2, 6)$, $(4, 2, 6, 4)$

   (b) $(3, 0, -3, 6)$, $(0, 2, 3, 1)$, $(0, -2, -2, 0)$, $(-2, 1, 2, 1)$

4. In each part, determine whether the vectors are linearly independent or are linearly dependent in P_2.

(a) $2 - x + 4x^2,\ 3 + 6x + 2x^2,\ 2 + 10x - 4x^2$

(b) $1 + 3x + 3x^2,\ x + 4x^2,\ 5 + 6x + 3x^2,\ 7 + 2x - x^2$

5. In each part, determine whether the matrices are linearly independent or dependent.

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ in M_{23}

6. Determine all values of k for which the following matrices are linearly independent in M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix},\ \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix},\ \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

7. In each part, determine whether the three vectors lie in a plane in R^3.

(a) $\mathbf{v}_1 = (2, -2, 0),\ \mathbf{v}_2 = (6, 1, 4),\ \mathbf{v}_3 = (2, 0, -4)$

(b) $\mathbf{v}_1 = (-6, 7, 2),\ \mathbf{v}_2 = (3, 2, 4),\ \mathbf{v}_3 = (4, -1, 2)$

8. In each part, determine whether the three vectors lie on the same line in R^3.

(a) $\mathbf{v}_1 = (-1, 2, 3),\ \mathbf{v}_2 = (2, -4, -6),\ \mathbf{v}_3 = (-3, 6, 0)$

(b) $\mathbf{v}_1 = (2, -1, 4),\ \mathbf{v}_2 = (4, 2, 3),\ \mathbf{v}_3 = (2, 7, -6)$

(c) $\mathbf{v}_1 = (4, 6, 8),\ \mathbf{v}_2 = (2, 3, 4),\ \mathbf{v}_3 = (-2, -3, -4)$

9. (a) Show that the three vectors $\mathbf{v}_1 = (0, 3, 1, -1)$, $\mathbf{v}_2 = (6, 0, 5, 1)$, and $\mathbf{v}_3 = (4, -7, 1, 3)$ form a linearly dependent set in R^4.

(b) Express each vector in part (a) as a linear combination of the other two.

10. (a) Show that the vectors $\mathbf{v}_1 = (1, 2, 3, 4)$, $\mathbf{v}_2 = (0, 1, 0, -1)$, and $\mathbf{v}_3 = (1, 3, 3, 3)$ form a linearly dependent set in R^4.

(b) Express each vector in part (a) as a linear combination of the other two.

11. For which real values of λ do the following vectors form a linearly dependent set in R^3?

$$\mathbf{v}_1 = \left(\lambda, -\tfrac{1}{2}, -\tfrac{1}{2}\right),\quad \mathbf{v}_2 = \left(-\tfrac{1}{2}, \lambda, -\tfrac{1}{2}\right),\quad \mathbf{v}_3 = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, \lambda\right)$$

12. Under what conditions is a set with one vector linearly independent?

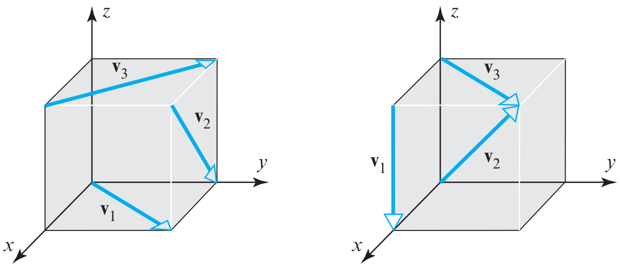
13. In each part, let $T_A\colon R^2 \to R^2$ be multiplication by A, and let $\mathbf{u}_1 = (1, 2)$ and $\mathbf{u}_2 = (-1, 1)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2)\}$ is linearly independent in R^2.

(a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

14. In each part, let $T_A\colon R^3 \to R^3$ be multiplication by A, and let $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (2, -1, 1)$, and $\mathbf{u}_3 = (0, 1, 1)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly independent in R^3.

(a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

15. Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in part (a) of the accompanying figure linearly independent? What about those in part (b)? Explain.



(a) (b)

▲ Figure Ex-15

16. By using appropriate identities, where required, determine which of the following sets of vectors in $F(-\infty, \infty)$ are linearly dependent.

(a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

(b) $x,\ \cos x$

(c) $1,\ \sin x,\ \sin 2x$

(d) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

(e) $(3 - x)^2,\ x^2 - 6x,\ 5$

(f) $0,\ \cos^3 \pi x,\ \sin^5 3\pi x$

17. (***Calculus required***) The functions

$$f_1(x) = x \quad \text{and} \quad f_2(x) = \cos x$$

are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using the Wronskian.

18. (***Calculus required***) The functions

$$f_1(x) = \sin x \quad \text{and} \quad f_2(x) = \cos x$$

are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using the Wronskian.

19. (***Calculus required***) Use the Wronskian to show that the following sets of vectors are linearly independent.

(a) $1,\ x,\ e^x$

(b) $1,\ x,\ x^2$

20. (***Calculus required***) Use the Wronskian to show that the functions $f_1(x) = e^x$, $f_2(x) = xe^x$, and $f_3(x) = x^2e^x$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

21. (***Calculus required***) Use the Wronskian to show that the functions $f_1(x) = \sin x$, $f_2(x) = \cos x$, and $f_3(x) = x\cos x$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

22. Show that for any vectors **u**, **v**, and **w** in a vector space V, the vectors $\mathbf{u}-\mathbf{v}$, $\mathbf{v}-\mathbf{w}$, and $\mathbf{w}-\mathbf{u}$ form a linearly dependent set.

23. (a) In Example 1 we showed that the mutually orthogonal vectors **i**, **j**, and **k** form a linearly independent set of vectors in R^3. Do you think that every set of three nonzero mutually orthogonal vectors in R^3 is linearly independent? Justify your conclusion with a geometric argument.

(b) Justify your conclusion with an algebraic argument. [*Hint:* Use dot products.]

Working with Proofs

24. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors, then so are $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1\}$, $\{\mathbf{v}_2\}$, and $\{\mathbf{v}_3\}$.

25. Prove that if $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

26. Prove that if $S=\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in a vector space V, and $\mathbf{v}_4$ is any vector in V that is not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

27. Prove that if $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set of vectors in a vector space V, and if $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ are any vectors in V that are not in S, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is also linearly dependent.

28. Prove that in P_2 every set with more than three vectors is linearly dependent.

29. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ does not lie in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

30. Use part (*a*) of Theorem 4.3.1 to prove part (*b*).

31. Prove part (*b*) of Theorem 4.3.2.

32. Prove part (*c*) of Theorem 4.3.2.

True-False Exercises

TF. In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) A set containing a single vector is linearly independent.

(b) The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar k.

(c) Every linearly dependent set contains the zero vector.

(d) If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ is also linearly independent for every nonzero scalar k.

(e) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.

(f) The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in M_{22}.

(g) The three polynomials $(x-1)(x+2)$, $x(x+2)$, and $x(x-1)$ are linearly independent.

(h) The functions f_1 and f_2 are linearly dependent if there is a real number x such that $k_1 f_1(x)+k_2 f_2(x)=0$ for some scalars k_1 and k_2.

Working with Technology

T1. Devise three different methods for using your technology utility to determine whether a set of vectors in R^n is linearly independent, and then use each of those methods to determine whether the following vectors are linearly independent.

$$\mathbf{v}_1=(4,-5,2,6),\quad \mathbf{v}_2=(2,-2,1,3),$$
$$\mathbf{v}_3=(6,-3,3,9),\quad \mathbf{v}_4=(4,-1,5,6)$$

T2. Show that $S=\{\cos t, \sin t, \cos 2t, \sin 2t\}$ is a linearly independent set in $C(-\infty,\infty)$ by evaluating the left side of the equation

$$c_1\cos t + c_2\sin t + c_3\cos 2t + c_4 \sin 2t = 0$$

at sufficiently many values of t to obtain a linear system whose only solution is $c_1=c_2=c_3=c_4=0$.

4.4 Coordinates and Basis

We usually think of a line as being one-dimensional, a plane as two-dimensional, and the space around us as three-dimensional. It is the primary goal of this section and the next to make this intuitive notion of dimension precise. In this section we will discuss coordinate systems in general vector spaces and lay the groundwork for a precise definition of dimension in the next section.

Coordinate Systems in Linear Algebra

In analytic geometry one uses *rectangular* coordinate systems to create a one-to-one correspondence between points in 2-space and ordered pairs of real numbers and between points in 3-space and ordered triples of real numbers (Figure 4.4.1). Although rectangular coordinate systems are common, they are not essential. For example, Figure 4.4.2 shows coordinate systems in 2-space and 3-space in which the coordinate axes are not mutually perpendicular.