$\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$	Hypothesis
$(\mathbf{u}+\mathbf{w})+(-\mathbf{w})=(\mathbf{v}+\mathbf{w})+(-\mathbf{w})$	Add $-\mathbf{w}$ to both sides.
$\mathbf{u}+[\mathbf{w}+(-\mathbf{w})]=\mathbf{v}+[\mathbf{w}+(-\mathbf{w})]$	____________
$\mathbf{u}+\mathbf{0}=\mathbf{v}+\mathbf{0}$	____________
$\mathbf{u}=\mathbf{v}$	____________

22. Below is a seven-step proof of part (*b*) of Theorem 4.1.1. Justify each step either by stating that it is true by *hypothesis* or by specifying which of the ten vector space axioms applies.

Hypothesis: Let $\mathbf{u}$ be any vector in a vector space V, let $\mathbf{0}$ be the zero vector in V, and let k be a scalar.

Conclusion: Then $k\mathbf{0}=\mathbf{0}$.

Proof: (1) $k\mathbf{0}+k\mathbf{u}=k(\mathbf{0}+\mathbf{u})$

(2) $\quad = k\mathbf{u}$

(3) Since $k\mathbf{u}$ is in V, $-k\mathbf{u}$ is in V.

(4) Therefore, $(k\mathbf{0}+k\mathbf{u})+(-k\mathbf{u})=k\mathbf{u}+(-k\mathbf{u})$.

(5) $k\mathbf{0}+(k\mathbf{u}+(-k\mathbf{u}))=k\mathbf{u}+(-k\mathbf{u})$

(6) $k\mathbf{0}+\mathbf{0}=\mathbf{0}$

(7) $k\mathbf{0}=\mathbf{0}$

▶ In Exercises 23–24, let $\mathbf{u}$ be any vector in a vector space V. Give a step-by-step proof of the stated result using Exercises 21 and 22 as models for your presentation. ◀

23. $0\mathbf{u}=\mathbf{0}$ **24.** $-\mathbf{u}=(-1)\mathbf{u}$

▶ In Exercises 25–27, prove that the given set with the stated operations is a vector space. ◀

25. The set $V=\{\mathbf{0}\}$ with the operations of addition and scalar multiplication given in Example 1.

26. The set R^∞ of all infinite sequences of real numbers with the operations of addition and scalar multiplication given in Example 3.

27. The set M_{mn} of all $m\times n$ matrices with the usual operations of addition and scalar multiplication.

28. Prove: If $\mathbf{u}$ is a vector in a vector space V and k a scalar such that $k\mathbf{u}=\mathbf{0}$, then either $k=0$ or $\mathbf{u}=\mathbf{0}$. [*Suggestion:* Show that if $k\mathbf{u}=\mathbf{0}$ and $k\neq 0$, then $\mathbf{u}=\mathbf{0}$. The result then follows as a logical consequence of this.]

True-False Exercises

TF. In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) A vector is any element of a vector space.

(b) A vector space must contain at least two vectors.

(c) If $\mathbf{u}$ is a vector and k is a scalar such that $k\mathbf{u}=\mathbf{0}$, then it must be true that $k=0$.

(d) The set of positive real numbers is a vector space if vector addition and scalar multiplication are the usual operations of addition and multiplication of real numbers.

(e) In every vector space the vectors $(-1)\mathbf{u}$ and $-\mathbf{u}$ are the same.

(f) In the vector space $F(-\infty,\infty)$ any function whose graph passes through the origin is a zero vector.

4.2 Subspaces

V vector space

10 Axioms

$W \subseteq V$

W subspace of V

It is often the case that some vector space of interest is contained within a larger vector space whose properties are known. In this section we will show how to recognize when this is the case, we will explain how the properties of the larger vector space can be used to obtain properties of the smaller vector space, and we will give a variety of important examples.

We begin with some terminology.

DEFINITION 1 A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In general, to show that a nonempty set W with two operations is a vector space one must verify the ten vector space axioms. However, if W is a subspace of a known vector space V, then certain axioms need not be verified because they are "inherited" from V. For example, it is *not* necessary to verify that $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ holds in W because it holds for all vectors in V including those in W. On the other hand, it *is* necessary to verify that W is closed under addition and scalar multiplication since it is possible that adding two vectors in W or multiplying a vector in W by a scalar produces a vector in V that is outside of W (Figure 4.2.1). Those axioms that are *not* inherited by W are

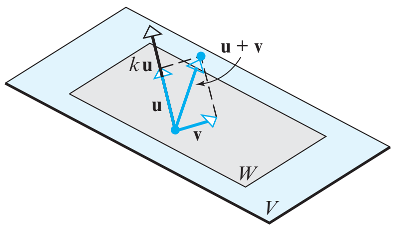
Axiom 1—Closure of W under addition

Axiom 4—Existence of a zero vector in W

Axiom 5—Existence of a negative in W for every vector in W

Axiom 6—Closure of W under scalar multiplication

so these must be verified to prove that it is a subspace of V. However, the next theorem shows that if Axiom 1 and Axiom 6 hold in W, then Axioms 4 and 5 hold in W as a consequence and hence need not be verified.



▶ Figure 4.2.1 The vectors **u** and **v** are in W, but the vectors $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ are not.

THEOREM 4.2.1 *If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.*

(*a*) *If* **u** *and* **v** *are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.* A1

(*b*) *If k is a scalar and* **u** *is a vector in W, then $k\mathbf{u}$ is in W.* A6

Proof If W is a subspace of V, then all the vector space axioms hold in W, including Axioms 1 and 6, which are precisely conditions (*a*) and (*b*).

Conversely, assume that conditions (*a*) and (*b*) hold. Since these are Axioms 1 and 6, and since Axioms 2, 3, 7, 8, 9, and 10 are inherited from V, we only need to show that Axioms 4 and 5 hold in W. For this purpose, let **u** be any vector in W. It follows from condition (*b*) that $k\mathbf{u}$ is a vector in W for every scalar k. In particular, $0\mathbf{u} = \mathbf{0}$ and $(-1)\mathbf{u} = -\mathbf{u}$ are in W, which shows that Axioms 4 and 5 hold in W. ◀

Theorem 4.2.1 states that W is a subspace of V if and only if it is closed under addition and scalar multiplication.

Note that every vector space has at least two subspaces, itself and its zero subspace.

▶ **EXAMPLE 1 The Zero Subspace** V , $W=\{0\}$

If V is any vector space, and if $W = \{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

$$\mathbf{0}+\mathbf{0}=\mathbf{0} \quad \text{and} \quad k\mathbf{0}=\mathbf{0}$$

for any scalar k. We call W the ***zero subspace*** of V.

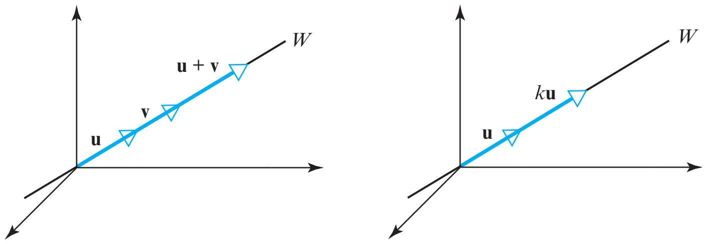
▶ **EXAMPLE 2 Lines Through the Origin Are Subspaces of R^2 and of R^3**

If W is a line through the origin of either R^2 or R^3, then adding two vectors on the line or multiplying a vector on the line by a scalar produces another vector on the line, so W is closed under addition and scalar multiplication (see Figure 4.2.2 for an illustration in R^3).

$V = R^2$ or R^3

(x,y) (x,y,z)

$x,y \in R$ $\in R$



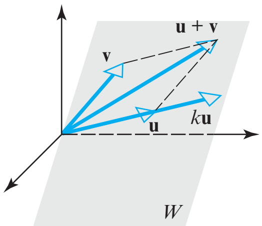
(*a*) W is closed under addition. (*b*) W is closed under scalar multiplication.

▶ Figure 4.2.2

$W=$

▶ **EXAMPLE 3 Planes Through the Origin Are Subspaces of R^3** $=V$

If **u** and **v** are vectors in a plane W through the origin of R^3, then it is evident geometrically that $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ also lie in the same plane W for any scalar k (Figure 4.2.3). Thus W is closed under addition and scalar multiplication. ◀



▲ Figure 4.2.3 The vectors $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ both lie in the same plane as **u** and **v**.

Table 1 below gives a list of subspaces of R^2 and of R^3 that we have encountered thus far. We will see later that these are the only subspaces of R^2 and of R^3.

Table 1

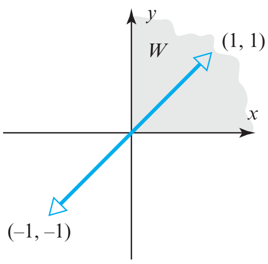
Subspaces of R^2	Subspaces of R^3
• $\{\mathbf{0}\}$	• $\{\mathbf{0}\}$
• Lines through the origin	• Lines through the origin
• R^2	• Planes through the origin
	• R^3

$V=$

▶ **EXAMPLE 4 A Subset of R^2 That Is Not a Subspace**

Let W be the set of all points (x, y) in R^2 for which $x \geq 0$ and $y \geq 0$ (the shaded region in Figure 4.2.4). This set is not a subspace of R^2 because it is not closed under scalar multiplication. For example, $\mathbf{v} = (1, 1)$ is a vector in W, but $(-1)\mathbf{v} = (-1, -1)$ is not.

$\in W$ $k=-1$ $\notin W$



▲ Figure 4.2.4 W is not closed under scalar multiplication.

▶ **EXAMPLE 5 Subspaces of M_{nn}**

We know from Theorem 1.7.2 that the sum of two symmetric $n \times n$ matrices is symmetric and that a scalar multiple of a symmetric $n \times n$ matrix is symmetric. Thus, the set of symmetric $n \times n$ matrices is closed under addition and scalar multiplication and hence is a subspace of M_{nn}. Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of M_{nn}.

W_1 W_2 W_3 W_4

invertible $\Leftrightarrow$ $\det A \neq 0$

▶ **EXAMPLE 6 A Subset of M_{nn} That Is Not a Subspace**

The set W of invertible $n \times n$ matrices is not a subspace of M_{nn}, failing on two counts—it is not closed under addition and not closed under scalar multiplication. We will illustrate this with an example in M_{22} that you can readily adapt to M_{nn}. Consider the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

$\det U = (1)(5) - (2)(2)$
$= 5 - 4 = 1 \neq 0$
U invertible

The matrix $0U$ is the 2×2 zero matrix and hence is not invertible, and the matrix $U + V$ has a column of zeros so it also is not invertible.

$\det V = (-1)(5) - (2)(-2)$
$= -5 + 4 = -1 \neq 0$
V invertible

$\Longrightarrow U+V = \begin{bmatrix} 0 & 4 \\ 0 & 10 \end{bmatrix}$

$\det(U+V)$
$= (0)(0) - (4)(0)$
$= 0 - 0 = 0$
not invertible

F(−∞, ∞) vector space

CALCULUS REQUIRED

EXAMPLE 7 The Subspace $C(-\infty, \infty)$

Continuous

There is a theorem in calculus which states that a sum of continuous functions is continuous and that a constant times a continuous function is continuous. Rephrased in vector language, the set of continuous functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C(-\infty, \infty)$.

CALCULUS REQUIRED

EXAMPLE 8 Functions with Continuous Derivatives

A function with a continuous derivative is said to be *continuously differentiable*. There is a theorem in calculus which states that the sum of two continuously differentiable functions is continuously differentiable and that a constant times a continuously differentiable function is continuously differentiable. Thus, the functions that are continuously differentiable on $(-\infty, \infty)$ form a subspace of $F(-\infty, \infty)$. We will denote this subspace by $C^1(-\infty, \infty)$, where the superscript emphasizes that the *first* derivatives are continuous. To take this a step further, the set of functions with m continuous derivatives on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$ as is the set of functions with derivatives of all orders on $(-\infty, \infty)$. We will denote these subspaces by $C^m(-\infty, \infty)$ and $C^\infty(-\infty, \infty)$, respectively.

EXAMPLE 9 The Subspace of All Polynomials

Recall that a ***polynomial*** is a function that can be expressed in the form

$$p(x) = a_0 + a_1x + \cdots + a_nx^n \tag{1}$$

where $a_0, a_1, \dots, a_n$ are constants. It is evident that the sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial. Thus, the set W of all polynomials is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_∞.

In this text we regard all constants to be polynomials of degree zero. Be aware, however, that some authors do not assign a degree to the constant 0.

EXAMPLE 10 The Subspace of Polynomials of Degree $\le n$

degree of n

Recall that the ***degree*** of a polynomial is the highest power of the variable that occurs with a nonzero coefficient. Thus, for example, if $a_n \neq 0$ in Formula (1), then that polynomial has degree n. It is *not* true that the set W of polynomials with positive degree n is a subspace of $F(-\infty, \infty)$ because that set is not closed under addition. For example, the polynomials

$$1 + 2x + 3x^2 \quad \text{and} \quad 5 + 7x - 3x^2 \Rightarrow 6+9x$$

both have degree 2, but their sum has degree 1. What *is* true, however, is that for each nonnegative integer n the polynomials of degree n *or less* form a subspace of $F(-\infty, \infty)$. We will denote this space by P_n.

degree of n or less

The Hierarchy of Function Spaces

It is proved in calculus that polynomials are continuous functions and have continuous derivatives of all orders on $(-\infty, \infty)$. Thus, it follows that P_∞ is not only a subspace of $F(-\infty, \infty)$, as previously observed, but is also a subspace of $C^\infty(-\infty, \infty)$. We leave it for you to convince yourself that the vector spaces discussed in Examples 7 to 10 are "nested" one inside the other as illustrated in Figure 4.2.5.

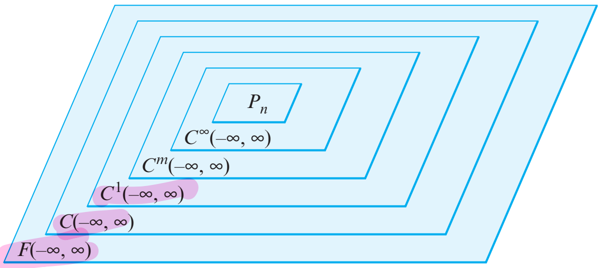
Remark In our previous examples we considered functions that were defined at all points of the interval $(-\infty, \infty)$. Sometimes we will want to consider functions that are only defined on some subinterval of $(-\infty, \infty)$, say the closed interval $[a, b]$ or the open interval (a, b). In such cases we will make an appropriate notation change. For example, $C[a, b]$ is the space of continuous functions on $[a, b]$ and $C(a, b)$ is the space of continuous functions on (a, b).

$V = \{ \quad \}$ vector spaces

$W = \{ \quad \}$ Is W subspace or not

① ①$\in W$ + ②$\in W$ = $\in W$ ✓

② k ①$\in W$ = k① $\in W$ ✓



▶ Figure 4.2.5

$\notin$ not subspace

Building Subspaces

The following theorem provides a useful way of creating a new subspace from known subspaces.

THEOREM 4.2.2 *If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.*

Proof Let W be the intersection of the subspaces $W_1, W_2, \ldots, W_r$. This set is not empty because each of these subspaces contains the zero vector of V, and hence so does their intersection. Thus, it remains to show that W is closed under addition and scalar multiplication.

To prove closure under addition, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in W. Since W is the intersection of $W_1, W_2, \ldots, W_r$, it follows that $\mathbf{u}$ and $\mathbf{v}$ also lie in each of these subspaces. Moreover, since these subspaces are closed under addition and scalar multiplication, they also all contain the vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for every scalar k, and hence so does their intersection W. This proves that W is closed under addition and scalar multiplication. ◀

Note that the first step in proving Theorem 4.2.2 was to establish that W contained at least one vector. This is important, for otherwise the subsequent argument might be logically correct but meaningless.

Sometimes we will want to find the "smallest" subspace of a vector space V that contains all of the vectors in some set of interest. The following definition, which generalizes Definition 4 of Section 3.1, will help us to do that.

If $k = 1$, then Equation (2) has the form $\mathbf{w} = k_1\mathbf{v}_1$, in which case the linear combination is just a scalar multiple of $\mathbf{v}_1$.

DEFINITION 2 If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{2}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

$V = \{v_1, v_2, \ldots, v_r\}$ w

$w = k_1v_1 + k_2v_2 + \cdots + k_rv_r$

THEOREM 4.2.3 *If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:*

$S \subseteq V$, W

(a) *The set W of all possible linear combinations of the vectors in S is a subspace of V.*

(b) *The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.*

$H \subseteq V$, $W \subseteq V \Rightarrow W \subseteq H$

Proof (a) Let W be the set of all possible linear combinations of the vectors in S. We must show that W is closed under addition and scalar multiplication. To prove closure under addition, let

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r \quad \text{and} \quad \mathbf{v} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r$$

be two vectors in W. It follows that their sum can be written as

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{w}_1 + (c_2 + k_2)\mathbf{w}_2 + \cdots + (c_r + k_r)\mathbf{w}_r$$

which is a linear combination of the vectors in S. Thus, W is closed under addition. We leave it for you to prove that W is also closed under scalar multiplication and hence is a subspace of V.

Proof (b) Let W' be any subspace of V that contains all of the vectors in S. Since W' is closed under addition and scalar multiplication, it contains all linear combinations of the vectors in S and hence contains W. ◀

The following definition gives some important notation and terminology related to Theorem 4.2.3.

In the case where S is the empty set, it will be convenient to agree that $\text{span}(\varnothing) = \{\mathbf{0}\}$.

DEFINITION 3 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V ***generated*** by S, and we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ ***span*** W. We denote this subspace as

$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad W = \text{span}(S)$$

▶ **EXAMPLE 11 The Standard Unit Vectors Span R^n** $= (x_1, x_2, \ldots x_n)$

Recall that the standard unit vectors in R^n are

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

These vectors span R^n since every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Thus, for example, the vectors

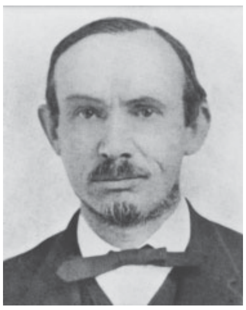
$$\underset{e_1}{\mathbf{i} = (1, 0, 0)}, \quad \underset{e_2}{\mathbf{j} = (0, 1, 0)}, \quad \underset{e_3}{\mathbf{k} = (0, 0, 1)}$$

span R^3 since every vector $\mathbf{v} = (a, b, c)$ in this space can be expressed as

$$\mathbf{v} = (a, b, c) = a\underset{k_1}{(1, 0, 0)} + b\underset{k_2}{(0, 1, 0)} + c\underset{k_3}{(0, 0, 1)} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

▶ **EXAMPLE 12 A Geometric View of Spanning in R^2 and R^3**

(a) If $\mathbf{v}$ is a nonzero vector in R^2 or R^3 that has its initial point at the origin, then $\text{span}\{\mathbf{v}\}$, which is the set of all scalar multiples of $\mathbf{v}$, is the line through the origin determined by $\mathbf{v}$. You should be able to visualize this from Figure 4.2.6a by observing that the tip of the vector $k\mathbf{v}$ can be made to fall at any point on the line by choosing the value of k to lengthen, shorten, or reverse the direction of $\mathbf{v}$ appropriately.

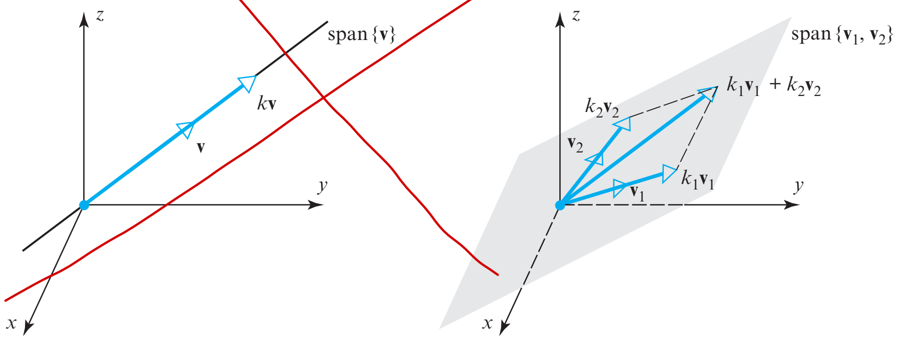


George William Hill (1838–1914)

Historical Note The term *linear combination* is due to the American mathematician G.W. Hill, who introduced it in a research paper on planetary motion published in 1900. Hill was a "loner" who preferred to work out of his home in West Nyack, New York, rather than in academia, though he did try lecturing at Columbia University for a few years. Interestingly, he apparently returned the teaching salary, indicating that he did not need the money and did not want to be bothered looking after it. Although technically a mathematician, Hill had little interest in modern developments of mathematics and worked almost entirely on the theory of planetary orbits.

[*Image: Courtesy of the American Mathematical Society www.ams.org*]

(b) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors in R^3 that have their initial points at the origin, then span$\{\mathbf{v}_1, \mathbf{v}_2\}$, which consists of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, is the plane through the origin determined by these two vectors. You should be able to visualize this from Figure 4.2.6*b* by observing that the tip of the vector $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$ can be made to fall at any point in the plane by adjusting the scalars k_1 and k_2 to lengthen, shorten, or reverse the directions of the vectors $k_1\mathbf{v}_1$ and $k_2\mathbf{v}_2$ appropriately.



(*a*) Span $\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$.

(*b*) Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

▶ Figure 4.2.6

▶ EXAMPLE 13 A Spanning Set for P_n

The polynomials $S=\{$ $1, x, x^2, \ldots, x^n$ span the vector space P_n defined in Example 10 since each polynomial $\mathbf{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. We can denote this by writing

$$P_n = \text{span}\{1, x, x^2, \ldots, x^n\}$$ ◀

$a_0 + a_1x + \cdots + a_nx^n = k_0 + k_1x + k_2x^2 + \cdots + k_nx^n$

find k_i

$k_0 = a_0$
$k_1 = a_1$
$k_2 = a_2$
$\vdots$
$k_n = a_n$

The next two examples are concerned with two important types of problems:

- Given a nonempty set S of vectors in R^n and a vector $\mathbf{v}$ in R^n, determine whether $\mathbf{v}$ is a linear combination of the vectors in S.
- Given a nonempty set S of vectors in R^n, determine whether the vectors span R^n.

▶ EXAMPLE 14 Linear Combinations

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is *not* a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution In order for $\mathbf{w}$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\boxed{\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}}$$

$(9,2,7) = k_1 u + k_2 v$
$= k_1(1,2,-1) + k_2(6,4,2)$
$(9,2,7) = (k_1+6k_2, 2k_1+4k_2, -k_1+2k_2)$

$8k_2 = 16$
$\boxed{k_2 = 2}$
$\boxed{k_1 = -3}$

Solving this system using Gaussian elimination yields $k_1 = -3$, $k_2 = 2$, so

$$\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$$

Similarly, for $\mathbf{w}'$ to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$, there must be scalars k_1 and k_2 such that $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{bmatrix}$$

$k_1 = -5$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

$8k_2 = 12$

$k_2 = \frac{3}{2}$

This system of equations is inconsistent (verify), so no such scalars k_1 and k_2 exist. Consequently, $\mathbf{w}'$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

EXAMPLE 15 Testing for Spanning

Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space R^3.

$S = \{v_1, v_2, v_3\}$

Solution We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Expressing this equation in terms of components gives

$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

or

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

$Ax = b$

or

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \phantom{{}+k_2} + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

Thus, our problem reduces to ascertaining whether this system is consistent for all values of b_1, b_2, and b_3. One way of doing this is to use parts (*e*) and (*g*) of Theorem 2.3.8, which state that the system is consistent if and only if its coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$\det(A) = -1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1(1) - 1(-1) = -1 + 1 = 0$

has a nonzero determinant. But this is *not* the case here since $\det(A) = 0$ (verify), so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3. ◀

Solution Spaces of Homogeneous Systems

The solutions of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns can be viewed as vectors in R^n. The following theorem provides a useful insight into the geometric structure of the solution set.

THEOREM 4.2.4 *The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.*

Proof Let W be the solution set of the system. The set W is not empty because it contains at least the trivial solution $\mathbf{x} = \mathbf{0}$.

To show that W is a subspace of R^n, we must show that it is closed under addition and scalar multiplication. To do this, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be vectors in W. Since these vectors are solutions of $A\mathbf{x} = \mathbf{0}$, we have

$$A\mathbf{x}_1 = \mathbf{0} \quad \text{and} \quad A\mathbf{x}_2 = \mathbf{0}$$

It follows from these equations and the distributive property of matrix multiplication that

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so W is closed under addition. Similarly, if k is any scalar then

$$A(k\mathbf{x}_1) = kA\mathbf{x}_1 = k\mathbf{0} = \mathbf{0}$$

so W is also closed under scalar multiplication. ◀

Because the solution set of a homogeneous system in n unknowns is actually a subspace of R^n, we will generally refer to it as the ***solution space*** of the system.

R^3

▶ EXAMPLE 16 Solution Spaces of Homogeneous Systems

In each part, solve the system by any method and then give a geometric description of the solution set.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$y + z = 0$

$x - 2y + 3z = 0$

let $z = t$

$y = -t$

$x = -5t$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$z = 0$

$y = 0$

$x = 0$

Solution

(a) The solutions are

$$x = 2s - 3t, \quad y = s, \quad z = t$$

$(\ , \ , \ )$

from which it follows that

$$x = 2y - 3z \quad \text{or} \quad x - 2y + 3z = 0$$

This is the equation of a plane through the origin that has $\mathbf{n} = (1, -2, 3)$ as a normal.

(b) The solutions are

$$x = -5t, \quad y = -t, \quad z = t$$

which are parametric equations for the line through the origin that is parallel to the vector $\mathbf{v} = (-5, -1, 1)$.

(c) The only solution is $x = 0$, $y = 0$, $z = 0$, so the solution space consists of the single point $\{\mathbf{0}\}$.

(d) This linear system is satisfied by all real values of x, y, and z, so the solution space is all of R^3. ◀

Remark Whereas the solution set of every *homogeneous* system of m equations in n unknowns is a subspace of R^n, it is *never* true that the solution set of a *nonhomogeneous* system of m equations in n unknowns is a subspace of R^n. There are two possible scenarios: first, the system may not have any solutions at all, and second, if there are solutions, then the solution set will not be closed either under addition or under scalar multiplication (Exercise 18).

$T_A : R^n \to R^m$

$Ax = 0$

The Linear Transformation Viewpoint

Theorem 4.2.4 can be viewed as a statement about matrix transformations by letting $T_A: R^n \to R^m$ be multiplication by the coefficient matrix A. From this point of view the solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in R^n that T_A maps into the zero vector in R^m. This set is sometimes called the ***kernel*** of the transformation, so with this terminology Theorem 4.2.4 can be rephrased as follows.

THEOREM 4.2.5 *If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A: R^n \to R^m$ is a subspace of R^n.*

A Concluding Observation

S.

R^n $(a, b, \ldots c)$

It is important to recognize that spanning sets are not unique. For example, any nonzero vector on the line in Figure 4.2.6*a* will span that line, and any two noncollinear vectors in the plane in Figure 4.2.6*b* will span that plane. The following theorem, whose proof is left as an exercise, states conditions under which two sets of vectors will span the same space.

THEOREM 4.2.6 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then*

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

Exercise Set 4.2

1. Use Theorem 4.2.1 to determine which of the following are subspaces of R^3.

(a) All vectors of the form $(a, 0, 0)$.

(b) All vectors of the form $(a, 1, 1)$.

(c) All vectors of the form (a, b, c), where $b = a + c$.

(d) All vectors of the form (a, b, c), where $b = a + c + 1$.

(e) All vectors of the form $(a, b, 0)$.

2. Use Theorem 4.2.1 to determine which of the following are subspaces of M_{nn}.

(a) The set of all diagonal $n \times n$ matrices.

(b) The set of all $n \times n$ matrices A such that $\det(A) = 0$.

(c) The set of all $n \times n$ matrices A such that $\text{tr}(A) = 0$.

(d) The set of all symmetric $n \times n$ matrices.

(e) The set of all $n \times n$ matrices A such that $A^T = -A$.

(f) The set of all $n \times n$ matrices A for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(g) The set of all $n \times n$ matrices A such that $AB = BA$ for some fixed $n \times n$ matrix B.

3. Use Theorem 4.2.1 to determine which of the following are subspaces of P_3.

(a) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

(b) All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.

(c) All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.

(d) All polynomials of the form $a_0 + a_1x$, where a_0 and a_1 are real numbers.

4. Which of the following are subspaces of $F(-\infty, \infty)$?

(a) All functions f in $F(-\infty, \infty)$ for which $f(0) = 0$.

(b) All functions f in $F(-\infty, \infty)$ for which $f(0) = 1$.

(c) All functions f in $F(-\infty, \infty)$ for which $f(-x) = f(x)$.

(d) All polynomials of degree 2.

5. Which of the following are subspaces of R^∞?

(a) All sequences $\mathbf{v}$ in R^∞ of the form $\mathbf{v} = (v, 0, v, 0, v, 0, \ldots)$.

(b) All sequences $\mathbf{v}$ in R^∞ of the form $\mathbf{v} = (v, 1, v, 1, v, 1, \ldots)$.

(c) All sequences $\mathbf{v}$ in R^∞ of the form $\mathbf{v} = (v, 2v, 4v, 8v, 16v, \ldots)$.

(d) All sequences in R^∞ whose components are 0 from some point on.

6. A line L through the origin in R^3 can be represented by parametric equations of the form $x = at$, $y = bt$, and $z = ct$. Use these equations to show that L is a subspace of R^3 by showing that if $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$ are points on L and k is any real number, then $k\mathbf{v}_1$ and $\mathbf{v}_1 + \mathbf{v}_2$ are also points on L.

7. Which of the following are linear combinations of $\mathbf{u} = (0, -2, 2)$ and $\mathbf{v} = (1, 3, -1)$?

(a) $(2, 2, 2)$ (b) $(0, 4, 5)$ (c) $(0, 0, 0)$

8. Express the following as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$.

(a) $(-9, -7, -15)$ (b) $(6, 11, 6)$ (c) $(0, 0, 0)$

9. Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}?$$

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

10. In each part express the vector as a linear combination of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$.

(a) $-9 - 7x - 15x^2$ (b) $6 + 11x + 6x^2$

(c) 0 (d) $7 + 8x + 9x^2$

11. In each part, determine whether the vectors span R^3.

(a) $\mathbf{v}_1 = (2, 2, 2)$, $\mathbf{v}_2 = (0, 0, 3)$, $\mathbf{v}_3 = (0, 1, 1)$

(b) $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (4, 1, 2)$, $\mathbf{v}_3 = (8, -1, 8)$

12. Suppose that $\mathbf{v}_1 = (2, 1, 0, 3)$, $\mathbf{v}_2 = (3, -1, 5, 2)$, and $\mathbf{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(a) $(2, 3, -7, 3)$ (b) $(0, 0, 0, 0)$

(c) $(1, 1, 1, 1)$ (d) $(-4, 6, -13, 4)$

13. Determine whether the following polynomials span P_2.

$$\mathbf{p}_1 = 1 - x + 2x^2, \quad \mathbf{p}_2 = 3 + x,$$
$$\mathbf{p}_3 = 5 - x + 4x^2, \quad \mathbf{p}_4 = -2 - 2x + 2x^2$$

14. Let $\mathbf{f} = \cos^2 x$ and $\mathbf{g} = \sin^2 x$. Which of the following lie in the space spanned by $\mathbf{f}$ and $\mathbf{g}$?

(a) $\cos 2x$ (b) $3 + x^2$ (c) 1 (d) $\sin x$ (e) 0

15. Determine whether the solution space of the system $A\mathbf{x} = \mathbf{0}$ is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it.

(a) $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}$

16. **(*Calculus required*)** Show that the following sets of functions are subspaces of $F(-\infty, \infty)$.

(a) All continuous functions on $(-\infty, \infty)$.

(b) All differentiable functions on $(-\infty, \infty)$.

(c) All differentiable functions on $(-\infty, \infty)$ that satisfy $\mathbf{f}' + 2\mathbf{f} = \mathbf{0}$.

17. **(*Calculus required*)** Show that the set of continuous functions $\mathbf{f} = f(x)$ on $[a, b]$ such that

$$\int_a^b f(x)\,dx = 0$$

is a subspace of $C[a, b]$.

18. Show that the solution vectors of a consistent nonhomogeneous system of m linear equations in n unknowns do not form a subspace of R^n.

19. In each part, let $T_A\colon R^2 \to R^2$ be multiplication by A, and let $\mathbf{u}_1 = (1, 2)$ and $\mathbf{u}_2 = (-1, 1)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2)\}$ spans R^2.

(a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

20. In each part, let $T_A\colon R^3 \to R^2$ be multiplication by A, and let $\mathbf{u}_1 = (0, 1, 1)$ and $\mathbf{u}_2 = (2, -1, 1)$ and $\mathbf{u}_3 = (1, 1, -2)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ spans R^2.

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix}$

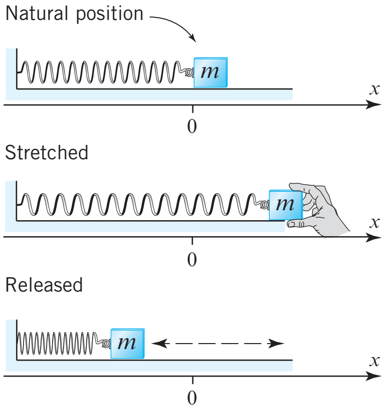
21. If T_A is multiplication by a matrix A with three columns, then the kernel of T_A is one of four possible geometric objects. What are they? Explain how you reached your conclusion.

22. Let $\mathbf{v}_1 = (1, 6, 4)$, $\mathbf{v}_2 = (2, 4, -1)$, $\mathbf{v}_3 = (-1, 2, 5)$, and $\mathbf{w}_1 = (1, -2, -5)$, $\mathbf{w}_2 = (0, 8, 9)$. Use Theorem 4.2.6 to show that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

23. The accompanying figure shows a mass-spring system in which a block of mass m is set into vibratory motion by pulling the block beyond its natural position at $x = 0$ and releasing it at time $t = 0$. If friction and air resistance are ignored, then the x-coordinate $x(t)$ of the block at time t is given by a function of the form

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

where ω is a fixed constant that depends on the mass of the block and the stiffness of the spring and c_1 and c_2 are arbitrary. Show that this set of functions forms a subspace of $C^\infty(-\infty, \infty)$.



▲ Figure Ex-23

Working with Proofs

24. Prove Theorem 4.2.6.

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Every subspace of a vector space is itself a vector space.

(b) Every vector space is a subspace of itself.

(c) Every subset of a vector space V that contains the zero vector in V is a subspace of V.

(d) The kernel of a matrix transformation $T_A: R^n \to R^m$ is a subspace of R^m.

(e) The solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns is a subspace of R^n.

(f) The span of any finite set of vectors in a vector space is closed under addition and scalar multiplication.

(g) The intersection of any two subspaces of a vector space V is a subspace of V.

(h) The union of any two subspaces of a vector space V is a subspace of V.

(i) Two subsets of a vector space V that span the same subspace of V must be equal.

(j) The set of upper triangular $n \times n$ matrices is a subspace of the vector space of all $n \times n$ matrices.

(k) The polynomials $x - 1$, $(x - 1)^2$, and $(x - 1)^3$ span P_3.

Working with Technology

T1. Recall from Theorem 1.3.1 that a product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of the matrix A in which the coefficients are the entries of $\mathbf{x}$. Use matrix multiplication to compute

$$\mathbf{v} = 6(8, -2, 1, -4) + 17(-3, 9, 11, 6) - 9(13, -1, 2, 4)$$

T2. Use the idea in Exercise T1 and matrix multiplication to determine whether the polynomial

$$\mathbf{p} = 1 + x + x^2 + x^3$$

is in the span of

$$\mathbf{p}_1 = 8 - 2x + x^2 - 4x^3, \quad \mathbf{p}_2 = -3 + 9x + 11x^2 + 6x^3,$$
$$\mathbf{p}_3 = 13 - x + 2x^2 + 4x^3$$

T3. For the vectors that follow, determine whether

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

$$\mathbf{v}_1 = (-1, 2, 0, 1, 3), \quad \mathbf{v}_2 = (7, 4, 6, -3, 1),$$
$$\mathbf{v}_3 = (-5, 3, 1, 2, 4)$$
$$\mathbf{w}_1 = (-6, 5, 1, 3, 7), \quad \mathbf{w}_2 = (6, 6, 6, -2, 4),$$
$$\mathbf{w}_3 = (2, 7, 7, -1, 5)$$

4.3 Linear Independence

In this section we will consider the question of whether the vectors in a given set are interrelated in the sense that one or more of them can be expressed as a linear combination of the others. This is important to know in applications because the existence of such relationships often signals that some kind of complication is likely to occur.

Linear Independence and Dependence

In a rectangular xy-coordinate system every vector in the plane can be expressed in exactly one way as a linear combination of the standard unit vectors. For example, the only way to express the vector $(3, 2)$ as a linear combination of $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ is

$$(3, 2) = 3(1, 0) + 2(0, 1) = 3\mathbf{i} + 2\mathbf{j} \tag{1}$$