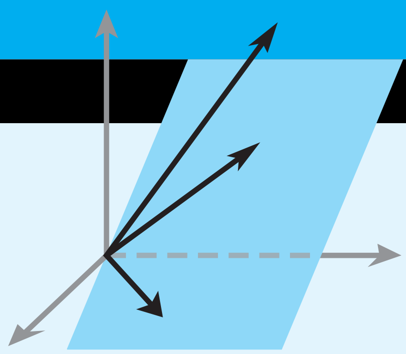

CHAPTER 2

Determinants

CHAPTER CONTENTS

INTRODUCTION In this chapter we will study "determinants" or, more precisely, "determinant functions." Unlike real-valued functions, such as $f(x) = x^2$, that assign a real number to a real variable x, determinant functions assign a real number $f(A)$ to a matrix variable A. Although determinants first arose in the context of solving systems of linear equations, they are rarely used for that purpose in real-world applications. While they can be useful for solving very small linear systems (say two or three unknowns), our main interest in them stems from the fact that they link together various concepts in linear algebra and provide a useful formula for the inverse of a matrix.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2.1 Determinants by Cofactor Expansion

In this section we will define the notion of a "determinant." This will enable us to develop a specific formula for the inverse of an invertible matrix, whereas up to now we have had only a computational procedure for finding it. This, in turn, will eventually provide us with a formula for solutions of certain kinds of linear systems.

$\neq 0$

$= 0$: A singular

Recall from Theorem 1.4.5 that the 2×2 matrix

$\det(A) = ad - bc$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$ and that the expression $ad - bc$ is called the ***determinant*** of the matrix A. Recall also that this determinant is denoted by writing

$$\det(A) = ad - bc \quad \text{or} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{1}$$

WARNING It is important to keep in mind that $\det(A)$ is a *number*, whereas A is a *matrix*.

and that the inverse of A can be expressed in terms of the determinant as

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

Minors and Cofactors

One of our main goals in this chapter is to obtain an analog of Formula (2) that is applicable to square matrices of *all orders*. For this purpose we will find it convenient to use subscripted entries when writing matrices or determinants. Thus, if we denote a 2×2 matrix as

M_{ij} C_{ij}

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then the two equations in (1) take the form

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{3}$$

In situations where it is inconvenient to assign a name to the matrix, we can express this formula as

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{4}$$

There are various methods for defining determinants of higher-order square matrices. In this text, we will us an "inductive definition" by which we mean that the determinant of a square matrix of a given order will be defined in terms of determinants of square matrices of the next lower order. To start the process, let us define the determinant of a 1×1 matrix $[a_{11}]$ as

$$\det[a_{11}] = a_{11} \tag{5}$$

from which it follows that Formula (4) can be expressed as

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det[a_{11}]\det[a_{22}] - \det[a_{12}]\det[a_{21}]$$

Now that we have established a starting point, we can define determinants of 3×3 matrices in terms of determinants of 2×2 matrices, then determinants of 4×4 matrices in terms of determinants of 3×3 matrices, and so forth, ad infinitum. The following terminology and notation will help to make this inductive process more efficient.

DEFINITION 1 If A is a square matrix, then the ***minor of entry*** $\boldsymbol{a_{ij}}$ is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the ***cofactor of entry*** $\boldsymbol{a_{ij}}$.

▶ EXAMPLE 1 Finding Minors and Cofactors

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

WARNING We have followed the standard convention of using capital letters to denote minors and cofactors even though they are numbers, not matrices.

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1}M_{11} = M_{11} = 16$$

Historical Note The term *determinant* was first introduced by the German mathematician Carl Friedrich Gauss in 1801 (see p. 15), who used them to "determine" properties of certain kinds of functions. Interestingly, the term *matrix* is derived from a Latin word for "womb" because it was viewed as a container of determinants.

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

M_{11} M_{32}

C_{11} C_{32}

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = (5)(8) - (4)(6) = 40 - 24 = \boxed{16}$$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2\, 16 = 16$$

$$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = (3)(6) - (2)(-4)$$

$$= 18 - (-8) = 18 + 8 = \boxed{26}$$

$$C_{32} = (-1)^{3+2} M_{32} = (-1)^5 \cdot 26 = \boxed{-26}$$

Similarly, the minor of entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

Remark Note that a minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the "checkerboard" array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For example,

$$C_{11} = M_{11}, \quad C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

and so forth. Thus, it is never really necessary to calculate $(-1)^{i+j}$ to calculate C_{ij}—you can simply compute the minor M_{ij} and then adjust the sign in accordance with the checkerboard pattern. Try this in Example 1.

$$A = \begin{bmatrix} \overset{+}{2} & \overset{-}{5} & \overset{+}{3} \\ \overset{-}{4} & \overset{+}{0} & \overset{-}{0} \\ \overset{+}{2} & \overset{-}{1} & \overset{+}{-5} \end{bmatrix}$$

EXAMPLE 2 Cofactor Expansions of a 2 × 2 Matrix

The checkerboard pattern for a 2×2 matrix $A = [a_{ij}]$ is

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

so that

$$\begin{aligned} C_{11} &= M_{11} = a_{22} & \qquad C_{12} &= -M_{12} = -a_{21} \\ C_{21} &= -M_{21} = -a_{12} & \qquad C_{22} &= M_{22} = a_{11} \end{aligned}$$

We leave it for you to use Formula (3) to verify that $\det(A)$ can be expressed in terms of cofactors in the following four ways:

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}C_{11} + a_{12}C_{12} \quad \leftarrow \text{row 1} \\ &= a_{21}C_{21} + a_{22}C_{22} \quad \leftarrow \text{row 2} \\ &= a_{11}C_{11} + a_{21}C_{21} \quad \leftarrow \text{column 1} \\ &= a_{12}C_{12} + a_{22}C_{22} \quad \leftarrow \text{column 2} \end{aligned} \tag{6}$$

Each of the last four equations is called a *cofactor expansion* of $\det(A)$. In each cofactor expansion the entries and cofactors all come from the same row or same column of A.

Historical Note The term *minor* is apparently due to the English mathematician James Sylvester (see p. 35), who wrote the following in a paper published in 1850: "Now conceive any one line and any one column be struck out, we get…a square, one term less in breadth and depth than the original square; and by varying in every possible selection of the line and column excluded, we obtain, supposing the original square to consist of n lines and n columns, n^2 such minor squares, each of which will represent what I term a "First Minor Determinant" relative to the principal or complete determinant."

For example, in the first equation the entries and cofactors all come from the first row of A, in the second they all come from the second row of A, in the third they all come from the first column of A, and in the fourth they all come from the second column of A. ◀

Definition of a General Determinant

Formula (6) is a special case of the following general result, which we will state without proof.

THEOREM 2.1.1 *If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.*

This result allows us to make the following definition.

DEFINITION 2 If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the ***determinant of A***, and the sums themselves are called ***cofactor expansions of A***. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{7}$$

[cofactor expansion along the *j*th column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{8}$$

[cofactor expansion along the *i*th row]

▶ **EXAMPLE 3 Cofactor Expansion Along the First Row**

Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

$$\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

Historical Note Cofactor expansion is not the only method for expressing the determinant of a matrix in terms of determinants of lower order. For example, although it is not well known, the English mathematician Charles Dodgson, who was the author of *Alice's Adventures in Wonderland* and *Through the Looking Glass* under the pen name of Lewis Carroll, invented such a method, called *condensation*. That method has recently been resurrected from obscurity because of its suitability for parallel processing on computers.

[*Image: Oscar G. Rejlander/ Time & Life Pictures/Getty Images*]

Charles Lutwidge Dodgson (Lewis Carroll) (1832–1898)

$$= 3(-4) - 1(-11) + 0 = -12 + 11 = -1$$

$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$

$= 3(-4) + 2(-2) + 5(3)$

$= -12 - 4 + 15$

$= -1$

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

▶ EXAMPLE 4 Cofactor Expansion Along the First Column

Let A be the matrix in Example 3, and evaluate $\det(A)$ by cofactor expansion along the first column of A.

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

This agrees with the result obtained in Example 3.

> Note that in Example 4 we had to compute three cofactors, whereas in Example 3 only two were needed because the third was multiplied by zero. As a rule, the best strategy for cofactor expansion is to expand along a row or column with the most zeros.

▶ EXAMPLE 5 Smart Choice of Row or Column

If A is the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

then to find $\det(A)$ it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For the 3×3 determinant, it will be easiest to use cofactor expansion along its second column, since it has the most zeros:

$$\det(A) = 1 \cdot -2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$
$$= -2(1+2)$$
$$= -6$$

▶ EXAMPLE 6 Determinant of a Lower Triangular Matrix

The following computation shows that the determinant of a 4×4 lower triangular matrix is the product of its diagonal entries. Each part of the computation uses a cofactor expansion along the first row.

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$
$$= a_{11}a_{22}\begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33}|a_{44}| = a_{11}a_{22}a_{33}a_{44} \blacktriangleleft$$

The method illustrated in Example 6 can be easily adapted to prove the following general result.

THEOREM 2.1.2 *If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.*

A Useful Technique for Evaluating 2×2 and 3×3 Determinants

Determinants of 2×2 and 3×3 matrices can be evaluated very efficiently using the pattern suggested in Figure 2.1.1.

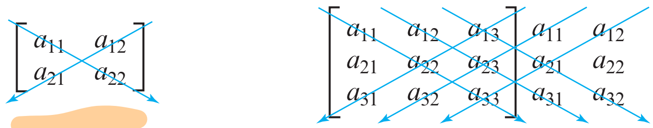

Figure 2.1.1

WARNING The arrow technique works only for determinants of 2×2 and 3×3 matrices. It *does not* work for matrices of size 4×4 or higher.

In the 2×2 case, the determinant can be computed by forming the product of the entries on the rightward arrow and subtracting the product of the entries on the leftward arrow. In the 3×3 case we first recopy the first and second columns as shown in the figure, after which we can compute the determinant by summing the products of the entries on the rightward arrows and subtracting the products on the leftward arrows. These procedures execute the computations

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}\end{aligned}$$

which agrees with the cofactor expansions along the first row.

EXAMPLE 7 A Technique for Evaluating 2 × 2 and 3 × 3 Determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{matrix}\right|\begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$ ◀

Ex 7

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (4)(1)$$

$$-6 - 4 = \boxed{-10}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{array}$$

$$= (1)(5)(9) + (2)(6)(7) + (3)(-4)(-8)$$

$$- \left[(2)(-4)(9) + (1)(6)(-8) + (3)(5)(7) \right]$$

$$= 45 + 84 + 96 + 72 + 48 - 105 = 240$$

Exercise Set 2.1

▶ In Exercises 1–2, find all the minors and cofactors of the matrix A. ◀

1. $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$ **2.** $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{bmatrix}$

3. Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Find

(a) M_{13} and C_{13}. (b) M_{23} and C_{23}.

(c) M_{22} and C_{22}. (d) M_{21} and C_{21}.

4. Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

Find

(a) M_{32} and C_{32}. (b) M_{44} and C_{44}.

(c) M_{41} and C_{41}. (d) M_{24} and C_{24}.

▶ In Exercises 5–8, evaluate the determinant of the given matrix. If the matrix is invertible, use Equation (2) to find its inverse. ◀

5. $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$ **7.** $\begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}$ **8.** $\begin{bmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{bmatrix}$

▶ In Exercises 9–14, use the arrow technique to evaluate the determinant. ◀

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$ **10.** $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix}$

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix}$ **12.** $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix}$

13. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix}$ **14.** $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

▶ In Exercises 15–18, find all values of λ for which $\det(A) = 0$. ◀

15. $A = \begin{bmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{bmatrix}$ **16.** $A = \begin{bmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{bmatrix}$

17. $A = \begin{bmatrix} \lambda-1 & 0 \\ 2 & \lambda+1 \end{bmatrix}$ **18.** $A = \begin{bmatrix} \lambda-4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-5 \end{bmatrix}$

19. Evaluate the determinant in Exercise 13 by a cofactor expansion along

(a) the first row. (b) the first column.

(c) the second row. (d) the second column.

(e) the third row. (f) the third column.

20. Evaluate the determinant in Exercise 12 by a cofactor expansion along

(a) the first row. (b) the first column.

(c) the second row. (d) the second column.

(e) the third row. (f) the third column.

▶ In Exercises 21–26, evaluate $\det(A)$ by a cofactor expansion along a row or column of your choice. ◀

21. $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$ **22.** $A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$ **24.** $A = \begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$

25. $A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$

26. $A = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix}$

▶ In Exercises 27–32, evaluate the determinant of the given matrix by inspection. ◀

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **28.** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

29. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & 2 & 3 & 8 \end{bmatrix}$ **30.** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ **32.** $\begin{bmatrix} -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 40 & 10 & -1 & 0 \\ 100 & 200 & -23 & 3 \end{bmatrix}$

33. In each part, show that the value of the determinant is independent of θ.

(a) $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$

(b) $\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$

34. Show that the matrices

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

35. By inspection, what is the relationship between the following determinants?

$$d_1 = \begin{vmatrix} a & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix} \quad \text{and} \quad d_2 = \begin{vmatrix} a+\lambda & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}$$

36. Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

37. What can you say about an nth-order determinant all of whose entries are 1? Explain.

38. What is the maximum number of zeros that a 3×3 matrix can have without having a zero determinant? Explain.

39. Explain why the determinant of a matrix with integer entries must be an integer.

Working with Proofs

40. Prove that (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear points if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

41. Prove that the equation of the line through the distinct points (a_1, b_1) and (a_2, b_2) can be written as

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

42. Prove that if A is upper triangular and B_{ij} is the matrix that results when the ith row and jth column of A are deleted, then B_{ij} is upper triangular if $i < j$.

True-False Exercises

TF. In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) The determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad + bc$.

(b) Two square matrices that have the same determinant must have the same size.

(c) The minor M_{ij} is the same as the cofactor C_{ij} if $i + j$ is even.

(d) If A is a 3×3 symmetric matrix, then $C_{ij} = C_{ji}$ for all i and j.

(e) The number obtained by a cofactor expansion of a matrix A is independent of the row or column chosen for the expansion.

(f) If A is a square matrix whose minors are all zero, then $\det(A) = 0$.

(g) The determinant of a lower triangular matrix is the sum of the entries along the main diagonal.

(h) For every square matrix A and every scalar c, it is true that $\det(cA) = c\det(A)$.

(i) For all square matrices A and B, it is true that

$$\det(A + B) = \det(A) + \det(B)$$

(j) For every 2×2 matrix A it is true that $\det(A^2) = (\det(A))^2$.

Working with Technology

T1. (a) Use the determinant capability of your technology utility to find the determinant of the matrix

$$A = \begin{bmatrix} 4.2 & -1.3 & 1.1 & 6.0 \\ 0.0 & 0.0 & -3.2 & 3.4 \\ 4.5 & 1.3 & 0.0 & 14.8 \\ 4.7 & 1.0 & 3.4 & 2.3 \end{bmatrix}$$

(b) Compare the result obtained in part (a) to that obtained by a cofactor expansion along the second row of A.

T2. Let A^n be the $n \times n$ matrix with 2's along the main diagonal, 1's along the diagonal lines immediately above and below the main diagonal, and zeros everywhere else. Make a conjecture about the relationship between n and $\det(A_n)$.