Working with Technology

T1. Starting with the formula stated in Exercise T1 of Section 1.5, derive a formula for the inverse of the "block diagonal" matrix

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

in which D_1 and D_2 are invertible, and use your result to compute the inverse of the matrix

$$M = \begin{bmatrix} 1.24 & 2.37 & 0 & 0 \\ 3.08 & -1.01 & 0 & 0 \\ 0 & 0 & 2.76 & 4.92 \\ 0 & 0 & 3.23 & 5.54 \end{bmatrix}$$

1.8 Matrix Transformations

In this section we will introduce a special class of functions that arise from matrix multiplication. Such functions, called "matrix transformations," are fundamental in the study of linear algebra and have important applications in physics, engineering, social sciences, and various branches of mathematics.

Recall that in Section 1.1 we defined an "ordered n-tuple" to be a sequence of n real numbers, and we observed that a solution of a linear system in n unknowns, say

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

can be expressed as the ordered n-tuple

$$(s_1, s_2, \ldots, s_n) \tag{1}$$

Recall also that if $n = 2$, then the n-tuple is called an "ordered pair," and if $n = 3$, it is called an "ordered triple." For two ordered n-tuples to be regarded as the same, they must list the same numbers in the same order. Thus, for example, (1, 2) and (2, 1) are different ordered pairs.

The set of all ordered n-tuples of real numbers is denoted by the symbol R^n. The elements of R^n are called ***vectors*** and are denoted in boldface type, such as **a**, **b**, **v**, **w**, and **x**. When convenient, ordered n-tuples can be denoted in matrix notation as column vectors. For example, the matrix

> The term "vector" is used in various ways in mathematics, physics, engineering, and other applications. The idea of viewing n-tuples as vectors will be discussed in more detail in Chapter 3, at which point we will also explain how this idea relates to more familiar notion of a vector.

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \tag{2}$$

can be used as an alternative to (1). We call (1) the ***comma-delimited form*** of a vector and (2) the ***column-vector form***. For each $i = 1, 2, \ldots, n$, let $\mathbf{e}_i$ denote the vector in R^n with a 1 in the ith position and zeros elsewhere. In column form these vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We call the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ the ***standard basis vectors*** for R^n. For example, the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are the standard basis vectors for R^3.

The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in R^n are termed "basis vectors" because all other vectors in R^n are expressible in exactly one way as a linear combination of them. For example, if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then we can express $\mathbf{x}$ as

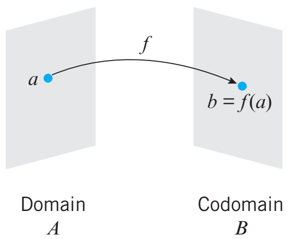
$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

Functions and Transformations

Recall that a ***function*** is a rule that associates with each element of a set A one and only one element in a set B. If f associates the element b with the element a, then we write

$$b = f(a)$$

and we say that b is the ***image*** of a under f or that $f(a)$ is the ***value*** of f at a. The set A is called the ***domain*** of f and the set B the ***codomain*** of f (Figure 1.8.1). The subset of the codomain that consists of all images of elements in the domain is called the ***range*** of f.



▲ Figure 1.8.1

In many applications the domain and codomain of a function are sets of real numbers, but in this text we will be concerned with functions for which the domain is R^n and the codomain is R^m for some positive integers m and n.

DEFINITION 1 If f is a function with domain R^n and codomain R^m, then we say that f is a ***transformation*** from R^n to R^m or that f ***maps*** from R^n to R^m, which we denote by writing

$$f: R^n \to R^m$$

In the special case where $m = n$, a transformation is sometimes called an ***operator*** on R^n.

$f: R^n \to R^m$

$(x_1, x_2, \ldots, x_n) \to (y_1, y_2, \ldots, y_m)$

Matrix Transformations

In this section we will be concerned with the class of transformations from R^n to R^m that arise from linear systems. Specifically, suppose that we have the system of linear equations

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\;\vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned} \tag{3}$$

which we can write in matrix notation as

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{4}$$

or more briefly as

$$\mathbf{w} = A\mathbf{x} \tag{5}$$

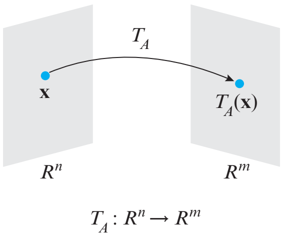
Although we could view (5) as a compact way of writing linear system (3), we will view it instead as a transformation that maps a vector $\mathbf{x}$ in R^n into thevector $\mathbf{w}$ in R^m by

It is common in linear algebra to use the letter T to denote a transformation. In keeping with this usage, we will usually denote a transformation from R^n to R^m by writing

$$T: R^n \to R^m$$

multiplying **x** on the left by A. We call this a ***matrix transformation*** (or ***matrix operator*** in the special case where $m = n$). We denote it by

$$T_A\colon R^n \to R^m$$



▲ Figure 1.8.2

(see Figure 1.8.2). This notation is useful when it is important to make the domain and codomain clear. The subscript on T_A serves as a reminder that the transformation results from multiplying vectors in R^n by the matrix A. In situations where specifying the domain and codomain is not essential, we will express (4) as

$$\mathbf{w} = T_A(\mathbf{x}) \tag{6}$$

We call the transformation T_A ***multiplication by A***. On occasion we will find it convenient to express (6) in the schematic form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w} \tag{7}$$

which is read "T_A maps **x** into **w**."

▶ EXAMPLE 1 A Matrix Transformation from R^4 to R^3

The transformation from R^4 to R^3 defined by the equations

$$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\ w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= 5x_1 - x_2 + 4x_3 \end{aligned} \tag{8}$$

can be expressed in matrix form as

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from which we see that the transformation can be interpreted as multiplication by

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \tag{9}$$

Although the image under the transformation T_A of any vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

in R^4 could be computed directly from the defining equations in (8), we will find it preferable to use the matrix in (9). For example, if

$$\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$$

given

then it follows from (9) that

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

3×4 $\quad 4 \times 1 \Rightarrow 3 \times 1$

EXAMPLE 2 Zero Transformations

If 0 is the $m \times n$ zero matrix, then

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

so multiplication by zero maps every vector in R^n into the zero vector in R^m. We call T_0 the ***zero transformation*** from R^n to R^m.

EXAMPLE 3 Identity Operators

If I is the $n \times n$ identity matrix, then

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

so multiplication by I maps every vector in R^n to itself. We call T_I the ***identity operator*** on R^n. ◀

Properties of Matrix Transformations

The following theorem lists four basic properties of matrix transformations that follow from properties of matrix multiplication.

THEOREM 1.8.1 *For every matrix A the matrix transformation $T_A: R^n \to R^m$ has the following properties for all vectors* **u** *and* **v** *and for every scalar k:*

(*a*) $T_A(\mathbf{0}) = \mathbf{0}$

(*b*) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ **[Homogeneity property]**

(*c*) $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ **[Additivity property]**

(*d*) $T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

Proof All four parts are restatements of the following properties of matrix arithmetic given in Theorem 1.4.1:

$$A\mathbf{0} = \mathbf{0}, \quad A(k\mathbf{u}) = k(A\mathbf{u}), \quad A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(\mathbf{u}-\mathbf{v}) = A\mathbf{u} - A\mathbf{v}$$ ◀

It follows from parts (*b*) and (*c*) of Theorem 1.8.1 that a matrix transformation maps a linear combination of vectors in R^n into the corresponding linear combination of vectors in R^m in the sense that

$$T_A(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r) = k_1T_A(\mathbf{u}_1) + k_2T_A(\mathbf{u}_2) + \cdots + k_rT_A(\mathbf{u}_r) \tag{10}$$

Matrix transformations are not the only kinds of transformations. For example, if

$$\begin{aligned} w_1 &= x_1^2 + x_2^2 \\ w_2 &= x_1x_2 \end{aligned} \tag{11}$$

then there are no constants a, b, c, and d for which

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 \\ x_1x_2 \end{bmatrix}$$

so that the equations in (11) do not define a matrix transformation from R^2 to R^2.

This leads us to the following two questions.

Question 1. Are there algebraic properties of a transformation $T: R^n \to R^m$ that can be used to determine whether T is a matrix transformation?

Question 2. If we discover that a transformation $T: R^n \to R^m$ is a matrix transformation, how can we find a matrix for it?

The following theorem and its proof will provide the answers.

THEOREM 1.8.2 *$T: R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in R^n and for every scalar k:*

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ **[Homogeneity property]**

Proof If T is a matrix transformation, then properties (i) and (ii) follow respectively from parts (c) and (b) of Theorem 1.8.1.

Conversely, assume that properties (i) and (ii) hold. We must show that there exists an $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for every vector $\mathbf{x}$ in R^n. Recall that the derivation of Formula (10) used only the additivity and homogeneity properties of T_A. Since we are assuming that T has those properties, it must be true that

$$T(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r) = k_1T(\mathbf{u}_1) + k_2T(\mathbf{u}_2) + \cdots + k_rT(\mathbf{u}_r) \tag{12}$$

for all scalars $k_1, k_2, \ldots, k_r$ and all vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ in R^n. Let A be the matrix

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)] \tag{13}$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n. It follows from Theorem 1.3.1 that $A\mathbf{x}$ is a linear combination of the columns of A in which the successive coefficients are the entries $x_1, x_2, \ldots, x_n$ of $\mathbf{x}$. That is,

$$A\mathbf{x} = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n)$$

Using Formula (10) we can rewrite this as

$$A\mathbf{x} = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = T(\mathbf{x})$$

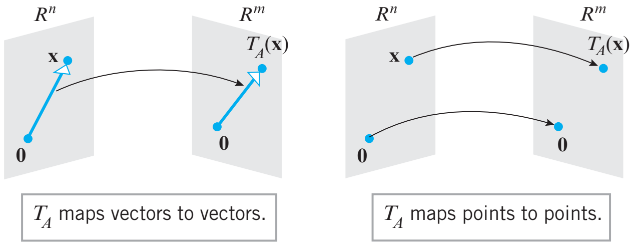
which completes the proof. ◄

The additivity and homogeneity properties in Theorem 1.8.2 are called ***linearity conditions***, and a transformation that satisfies these conditions is called a ***linear transformation***. Using this terminology Theorem 1.8.2 can be restated as follows.

Theorem 1.8.3 tells us that *for transformations from R^n to R^m*, the terms "matrix transformation" and "linear transformation" are synonymous.

THEOREM 1.8.3 *Every linear transformation from R^n to R^m is a matrix transformation, and conversely, every matrix transformation from R^n to R^m is a linear transformation.*

Depending on whether n-tuples and m-tuples are regarded as vectors or points, the geometric effect of a matrix transformation $T_A: R^n \to R^m$ is to map each vector (point) in R^n into a vector (point) in R^m (Figure 1.8.3).



▶ Figure 1.8.3

The following theorem states that if two matrix transformations from R^n to R^m have the same image at each point of R^n, then the matrices themselves must be the same.

THEOREM 1.8.4 *If $T_A: R^n \to R^m$ and $T_B: R^n \to R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.*

Proof To say that $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector in R^n is the same as saying that

$$A\mathbf{x} = B\mathbf{x}$$

for every vector $\mathbf{x}$ in R^n. This will be true, in particular, if $\mathbf{x}$ is any of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n; that is,

$$A\mathbf{e}_j = B\mathbf{e}_j \quad (j = 1, 2, \ldots, n) \tag{14}$$

Since every entry of $\mathbf{e}_j$ is 0 except for the jth, which is 1, it follows from Theorem 1.3.1 that $A\mathbf{e}_j$ is the jth column of A and $B\mathbf{e}_j$ is the jth column of B. Thus, (14) implies that corresponding columns of A and B are the same, and hence that $A = B$. ◀

Theorem 1.8.4 is significant because it tells us that there is a *one-to-one correspondence* between $m \times n$ matrices and matrix transformations from R^n to R^m in the sense that every $m \times n$ matrix A produces exactly one matrix transformation (multiplication by A) and every matrix transformation from R^n to R^m arises from exactly one $m \times n$ matrix; we call that matrix the ***standard matrix*** for the transformation.

A Procedure for Finding Standard Matrices

In the course of proving Theorem 1.8.2 we showed in Formula (13) that if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n (in column form), then the standard matrix for a linear transformation $T: R^n \to R^m$ is given by the formula

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)] \tag{15}$$

This suggests the following procedure for finding standard matrices.

Finding the Standard Matrix for a Matrix Transformation

Step 1. Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n.

Step 2. Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

R^n

$e_1 = (\underbrace{1, 0, 0, \ldots 0}_{n \text{ times}},)$, $e_2 = (0, 1, 0, \ldots\ldots 0)$ $e_n = (0, 0, \ldots, 1)$

$e_3 = (0, 0, 1, 0, \ldots 0)$

R^3: $e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

$e_1 = (1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (1)

$e_2 = (0,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (2)

EXAMPLE 4 Finding a Standard Matrix

Find the standard matrix A for the linear transformation $T: R^2 \to R^3$ defined by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix} \tag{16}$$

Solution We leave it for you to verify that

$x_1 = 1, x_2 = 0$ $\qquad$ $x_1 = 0, x_2 = 1$

(1) $$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Thus, it follows from Formulas (15) and (16) that the standard matrix is

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

EXAMPLE 5 Computing with Standard Matrices

For the linear transformation in Example 4, use the standard matrix A obtained in that example to find

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

Although we could have obtained the result in Example 5 by substituting values for the variables in (13), the method used in Example 5 is preferable for large-scale problems in that matrix multiplication is better suited for computer computations.

Solution The transformation is multiplication by A, so

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 3 \end{bmatrix} \blacktriangleleft$$

For transformation problems posed in comma-delimited form, a good procedure is to rewrite the problem in column-vector form and use the methods previously illustrated.

EXAMPLE 6 Finding a Standard Matrix

Rewrite the transformation $T(x_1, x_2) = (3x_1 + x_2, 2x_1 - 4x_2)$ in column-vector form and find its standard matrix.

Solution

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, the standard matrix is

$$\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \blacktriangleleft$$

Remark This section is but a first step in the study of linear transformations, which is one of the major themes in this text. We will delve deeper into this topic in Chapter 4, at which point we will have more background and a richer source of examples to work with.

Exercise Set 1.8

In Exercises 1–2, find the domain and codomain of the transformation $T_A(\mathbf{x}) = A\mathbf{x}$.

1. (a) A has size 3×2. (b) A has size 2×3.
(c) A has size 3×3. (d) A has size 1×6.

2. (a) A has size 4×5. (b) A has size 5×4.
(c) A has size 4×4. (d) A has size 3×1.

In Exercises 3–4, find the domain and codomain of the transformation defined by the equations.

3. (a) $\begin{aligned} w_1 &= 4x_1 + 5x_2 \\ w_2 &= x_1 - 8x_2 \end{aligned}$ (b) $\begin{aligned} w_1 &= 5x_1 - 7x_2 \\ w_2 &= 6x_1 + x_2 \\ w_3 &= 2x_1 + 3x_2 \end{aligned}$

4. (a) $\begin{aligned} w_1 &= x_1 - 4x_2 + 8x_3 \\ w_2 &= -x_1 + 4x_2 + 2x_3 \\ w_3 &= -3x_1 + 2x_2 - 5x_3 \end{aligned}$ (b) $\begin{aligned} w_1 &= 2x_1 + 7x_2 - 4x_3 \\ w_2 &= 4x_1 - 3x_2 + 2x_3 \end{aligned}$

In Exercises 5–6, find the domain and codomain of the transformation defined by the matrix product.

5. (a) $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

6. (a) $\begin{bmatrix} 6 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & -6 \\ 3 & 7 & -4 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

In Exercises 7–8, find the domain and codomain of the transformation T defined by the formula.

7. (a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$
(b) $T(x_1, x_2, x_3) = (4x_1 + x_2, x_1 + x_2)$

8. (a) $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$
(b) $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$

In Exercises 9–10, find the domain and codomain of the transformation T defined by the formula.

9. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 \\ x_1 - x_2 \\ 3x_2 \end{bmatrix}$ **10.** $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 - x_3 \\ 0 \end{bmatrix}$

In Exercises 11–12, find the standard matrix for the transformation defined by the equations.

11. (a) $\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 \\ w_2 &= 3x_1 + 5x_2 - x_3 \end{aligned}$ (b) $\begin{aligned} w_1 &= 7x_1 + 2x_2 - 8x_3 \\ w_2 &= \quad - x_2 + 5x_3 \\ w_3 &= 4x_1 + 7x_2 - x_3 \end{aligned}$

12. (a) $\begin{aligned} w_1 &= -x_1 + x_2 \\ w_2 &= 3x_1 - 2x_2 \\ w_3 &= 5x_1 - 7x_2 \end{aligned}$ (b) $\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 + x_2 \\ w_3 &= x_1 + x_2 + x_3 \\ w_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$

13. Find the standard matrix for the transformation T defined by the formula.
(a) $T(x_1, x_2) = (x_2, -x_1, x_1 + 3x_2, x_1 - x_2)$
(b) $T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$
(c) $T(x_1, x_2, x_3) = (0, 0, 0, 0, 0)$
(d) $T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_3, x_2, x_1 - x_3)$

14. Find the standard matrix for the operator T defined by the formula.
(a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$
(b) $T(x_1, x_2) = (x_1, x_2)$
(c) $T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 + 5x_2, x_3)$
(d) $T(x_1, x_2, x_3) = (4x_1, 7x_2, -8x_3)$

15. Find the standard matrix for the operator $T: R^3 \to R^3$ defined by

$$\begin{aligned} w_1 &= 3x_1 + 5x_2 - x_3 \\ w_2 &= 4x_1 - x_2 + x_3 \\ w_3 &= 3x_1 + 2x_2 - x_3 \end{aligned}$$

and then compute $T(-1, 2, 4)$ by directly substituting in the equations and then by matrix multiplication.

16. Find the standard matrix for the transformation $T: R^4 \to R^2$ defined by

$$\begin{aligned} w_1 &= 2x_1 + 3x_2 - 5x_3 - x_4 \\ w_2 &= x_1 - 5x_2 + 2x_3 - 3x_4 \end{aligned}$$

and then compute $T(1, -1, 2, 4)$ by directly substituting in the equations and then by matrix multiplication.

In Exercises 17–18, find the standard matrix for the transformation and use it to compute $T(\mathbf{x})$. Check your result by substituting directly in the formula for T.

17. (a) $T(x_1, x_2) = (-x_1 + x_2, x_2)$; $\mathbf{x} = (-1, 4)$
(b) $T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_2 + x_3, 0)$; $\mathbf{x} = (2, 1, -3)$

18. (a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$; $\mathbf{x} = (-2, 2)$
(b) $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$; $\mathbf{x} = (1, 0, 5)$

In Exercises 19–20, find $T_A(\mathbf{x})$, and express your answer in matrix form.

19. (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

20. (a) $A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

▶ In Exercises 21–22, use Theorem 1.8.2 to show that T is a matrix transformation. ◀

21. (a) $T(x, y) = (2x + y, x - y)$

(b) $T(x_1, x_2, x_3) = (x_1, x_3, x_1 + x_2)$

22. (a) $T(x, y, z) = (x + y, y + z, x)$

(b) $T(x_1, x_2) = (x_2, x_1)$

▶ In Exercises 23–24, use Theorem 1.8.2 to show that T is not a matrix transformation. ◀

23. (a) $T(x, y) = (x^2, y)$

(b) $T(x, y, z) = (x, y, xz)$

24. (a) $T(x, y) = (x, y + 1)$

(b) $T(x_1, x_2, x_3) = \left(x_1, x_2, \sqrt{x_3}\right)$

25. A function of the form $f(x) = mx + b$ is commonly called a "linear function" because the graph of $y = mx + b$ is a line. Is f a matrix transformation on R?

26. Show that $T(x, y) = (0, 0)$ defines a matrix operator on R^2 but $T(x, y) = (1, 1)$ does not.

▶ In Exercises 27–28, the images of the standard basis vectors for R^3 are given for a linear transformation $T: R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$. ◀

27. $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

28. $T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

29. Let $T: R^2 \to R^2$ be a linear operator for which the images of the standard basis vectors for R^2 are $T(\mathbf{e}_1) = (a, b)$ and $T(\mathbf{e}_2) = (c, d)$. Find $T(1, 1)$.

30. We proved in the text that if $T: R^n \to R^m$ is a matrix transformation, then $T(\mathbf{0}) = \mathbf{0}$. Show that the converse of this result is false by finding a mapping $T: R^n \to R^m$ that is not a matrix transformation but for which $T(\mathbf{0}) = \mathbf{0}$.

31. Let $T_A: R^3 \to R^3$ be multiplication by

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 2 \\ 4 & 5 & -3 \end{bmatrix}$$

and let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the standard basis vectors for R^3. Find the following vectors by inspection.

(a) $T_A(\mathbf{e}_1)$, $T_A(\mathbf{e}_2)$, and $T_A(\mathbf{e}_3)$

(b) $T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ (c) $T_A(7\mathbf{e}_3)$

Working with Proofs

32. (a) Prove: If $T: R^n \to R^m$ is a matrix transformation, then $T(\mathbf{0}) = \mathbf{0}$; that is, T maps the zero vector in R^n into the zero vector in R^m.

(b) The converse of this is not true. Find an example of a function T for which $T(\mathbf{0}) = \mathbf{0}$ but which is not a matrix transformation.

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) If A is a 2×3 matrix, then the domain of the transformation T_A is R^2.

(b) If A is an $m \times n$ matrix, then the codomain of the transformation T_A is R^n.

(c) There is at least one linear transformation $T: R^n \to R^m$ for which $T(2\mathbf{x}) = 4T(\mathbf{x})$ for some vector $\mathbf{x}$ in R^n.

(d) There are linear transformations from R^n to R^m that are not matrix transformations.

(e) If $T_A: R^n \to R^n$ and if $T_A(\mathbf{x}) = \mathbf{0}$ for every vector $\mathbf{x}$ in R^n, then A is the $n \times n$ zero matrix.

(f) There is only one matrix transformation $T: R^n \to R^m$ such that $T(-\mathbf{x}) = -T(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n.

(g) If $\mathbf{b}$ is a nonzero vector in R^n, then $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ is a matrix operator on R^n.