# 1.6 More on Linear Systems and Invertible Matrices

In this section we will show how the inverse of a matrix can be used to solve a linear system and we will develop some more results about invertible matrices.

## Number of Solutions of a Linear System

One many No

In Section 1.1 we made the statement (based on Figures 1.1.1 and 1.1.2) that every linear system either has no solutions, has exactly one solution, or has infinitely many solutions. We are now in a position to prove this fundamental result.

**THEOREM 1.6.1** *A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

***Proof*** If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations, exactly one of the following is true: (a) the system has no solutions, (b) the system has exactly one solution, or (c) the system has more than one solution. The proof will be complete if we can show that the system has infinitely many solutions in case (c).

Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, the matrix $\mathbf{x}_0$ is nonzero; moreover,

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

If we now let $k$ be any scalar, then

$$\begin{aligned} A(\mathbf{x}_1 + k\mathbf{x}_0) &= A\mathbf{x}_1 + A(k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0) \\ &= \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b} \end{aligned}$$

But this says that $\mathbf{x}_1 + k\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{x}_0$ is nonzero and there are infinitely many choices for $k$, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. ◀

## Solving Linear Systems by Matrix Inversion

$Ax = b$

$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Thus far we have studied two *procedures* for solving linear systems—Gauss–Jordan elimination and Gaussian elimination. The following theorem provides an actual *formula* for the solution of a linear system of $n$ equations in $n$ unknowns in the case where the coefficient matrix is invertible.

**THEOREM 1.6.2** *If $A$ is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

***Proof*** Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. To show that this is the only solution, we will assume that $\mathbf{x}_0$ is an arbitrary solution and then show that $\mathbf{x}_0$ must be the solution $A^{-1}\mathbf{b}$.

If $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x}_0 = \mathbf{b}$. Multiplying both sides of this equation by $A^{-1}$, we obtain $\mathbf{x}_0 = A^{-1}\mathbf{b}$. ◀

▶ **EXAMPLE 1 Solution of a Linear System Using $A^{-1}$**

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\; + 8x_3 &= 17 \end{aligned}$$

$A^{-1}Ax = A^{-1}b$

$I \cdot x = A^{-1}b$

$x = A^{-1}b$

$Ax = b$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$

$x = A^{-1} \cdot b$



$[A \mid I]$

$[I \mid A^{-1}]$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section, we showed that $A$ is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2, the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \underbrace{\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}}_{3\times3} \underbrace{\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}}_{3\times} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{3\times 1}$$

$x_1 = 1$
$x_2 = -1$
$x_3 = 2$

or $x_1 = 1, x_2 = -1, x_3 = 2$. ◀

$(x_1, x_2, x_3) = (1, -1, 2)$

Keep in mind that the method of Example 1 only applies when the system has as many equations as unknowns and the coefficient matrix is invertible.

**Linear Systems with a Common Coefficient Matrix**

$[A \mid b_1]$
$[A \mid b_2]$
$[A \mid b_3]$

Frequently, one is concerned with solving a sequence of systems

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \ldots, \quad A\mathbf{x} = \mathbf{b}_k$$

each of which has the same square coefficient matrix $A$. If $A$ is invertible, then the solutions

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1, \quad \mathbf{x}_2 = A^{-1}\mathbf{b}_2, \quad \mathbf{x}_3 = A^{-1}\mathbf{b}_3, \ldots, \quad \mathbf{x}_k = A^{-1}\mathbf{b}_k$$

can be obtained with one matrix inversion and $k$ matrix multiplications. An efficient way to do this is to form the partitioned matrix

$$[A \mid \mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_k] \tag{1}$$

in which the coefficient matrix $A$ is "augmented" by all $k$ of the matrices $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$, and then reduce (1) to reduced row echelon form by Gauss–Jordan elimination. In this way we can solve all $k$ systems at once. This method has the added advantage that it applies even when $A$ is not invertible.

**EXAMPLE 2 Solving Two Linear Systems at Once**

Solve the systems

(a) $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \phantom{+ 2x_2} + 8x_3 &= 9 \end{aligned}$ ($b_1$)

(b) $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \phantom{+ 2x_2} + 8x_3 &= -6 \end{aligned}$ ($b_2$)

***Solution*** The two systems have the same coefficient matrix. If we augment this coefficient matrix with the columns of constants on the right sides of these systems, we obtain

$$\begin{array}{ccccc} & & & b_1 & b_2 \end{array}$$
$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

Reducing this matrix to reduced row echelon form yields (verify)

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

the columns of constants on the right sides of

$$\begin{array}{c} \begin{array}{cccccc} & & & b_1 & b_2 \end{array} \\ \left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right] \end{array}$$

to reduced row echelon form yields (verify)

$$\xrightarrow[-1R1+R3]{-2R1+R2} \left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & -2 & 5 & 5 & -7 \end{array}\right]$$

$$\xrightarrow{2R3+R2} \left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-R3} \left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[-3R3+R1]{3R3+R2} \left[\begin{array}{ccc|c|c} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-2R2+R1} \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

(a) $x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$x_1 = 1$

$x_2 = 0$

$x_3 = 1$

(b) $x = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$x_1 = 2$

$x_2 = 1$

$x_3 = -1$

It follows from the last two columns that the solution of system (a) is $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ and the solution of system (b) is $x_1 = 2$, $x_2 = 1$, $x_3 = -1$. ◀

### Properties of Invertible Matrices

Up to now, to show that an $n \times n$ matrix $A$ is invertible, it has been necessary to find an $n \times n$ matrix $B$ such that

$$AB = I \quad \text{and} \quad BA = I$$

The next theorem shows that if we produce an $n \times n$ matrix $B$ satisfying *either* condition, then the other condition will hold automatically.

**THEOREM 1.6.3** *Let $A$ be a square matrix.*

(a) *If $B$ is a square matrix satisfying $BA = I$, then $B = A^{-1}$.*

(b) *If $B$ is a square matrix satisfying $AB = I$, then $B = A^{-1}$.*

$A^{-1}A = I$

$AA^{-1} = I$

We will prove part (*a*) and leave part (*b*) as an exercise.

**Proof (a)** Assume that $BA = I$. If we can show that $A$ is invertible, the proof can be completed by multiplying $BA = I$ on both sides by $A^{-1}$ to obtain

$$BAA^{-1} = IA^{-1} \quad \text{or} \quad BI = IA^{-1} \quad \text{or} \quad B = A^{-1}$$

To show that $A$ is invertible, it suffices to show that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (see Theorem 1.5.3). Let $\mathbf{x}_0$ be any solution of this system. If we multiply both sides of $A\mathbf{x}_0 = \mathbf{0}$ on the left by $B$, we obtain $BA\mathbf{x}_0 = B\mathbf{0}$ or $I\mathbf{x}_0 = \mathbf{0}$ or $\mathbf{x}_0 = \mathbf{0}$. Thus, the system of equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. ◀

### Equivalence Theorem

We are now in a position to add two more statements to the four given in Theorem 1.5.3.

**THEOREM 1.6.4 Equivalent Statements**

*If $A$ is an $n \times n$ matrix, then the following are equivalent.*

(a) *$A$ is invertible.*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(c) *The reduced row echelon form of $A$ is $I_n$.*

(d) *$A$ is expressible as a product of elementary matrices.*

(e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.*

(f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.*

Consistent

one many

**Proof** Since we proved in Theorem 1.5.3 that (*a*), (*b*), (*c*), and (*d*) are equivalent, it will be sufficient to prove that (*a*) ⇒ (*f*) ⇒ (*e*) ⇒ (*a*).

**(a) ⇒ (f)** This was already proved in Theorem 1.6.2.

**(f) ⇒ (e)** This is almost self-evident, for if $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$, then $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

**(e) ⇒ (a)** If the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$, then, in particular, this is so for the systems

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be solutions of the respective systems, and let us form an $n \times n$ matrix $C$ having these solutions as columns. Thus $C$ has the form

$$C = [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_n]$$

As discussed in Section 1.3, the successive columns of the product $AC$ will be

$$A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n$$

[see Formula (8) of Section 1.3]. Thus,

It follows from the equivalency of parts (*e*) and (*f*) that if you can show that $A\mathbf{x} = \mathbf{b}$ has at *least one* solution for every $n \times 1$ matrix $\mathbf{b}$, then you can conclude that it has *exactly one* solution for every $n \times 1$ matrix $\mathbf{b}$.

$$AC = [A\mathbf{x}_1 \mid A\mathbf{x}_2 \mid \cdots \mid A\mathbf{x}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

By part (*b*) of Theorem 1.6.3, it follows that $C = A^{-1}$. Thus, $A$ is invertible. ◀

We know from earlier work that invertible matrix factors produce an invertible product. Conversely, the following theorem shows that if the product of square matrices is invertible, then the factors themselves must be invertible.

sec 1.4
if A and B are invertiable then AB is invertiable

if P

**THEOREM 1.6.5** *Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.*

q

P⇐q

***Proof*** We will show first that $B$ is invertible by showing that the homogeneous system $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. If we assume that $\mathbf{x}_0$ is any solution of this system, then

$$(AB)\mathbf{x}_0 = A(B\mathbf{x}_0) = A\mathbf{0} = \mathbf{0}$$

so $\mathbf{x}_0 = \mathbf{0}$ by parts (*a*) and (*b*) of Theorem 1.6.4 applied to the invertible matrix $AB$. But the invertibility of $B$ implies the invertibility of $B^{-1}$ (Theorem 1.4.7), which in turn implies that

$$(AB)B^{-1} = A(BB^{-1}) = AI = A$$

is invertible since the left side is a product of invertible matrices. This completes the proof. ◀

In our later work the following fundamental problem will occur frequently in various contexts.

**A Fundamental Problem** Let $A$ be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices $\mathbf{b}$ such that the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent.

If $A$ is an invertible matrix, Theorem 1.6.2 completely solves this problem by asserting that for *every* $m \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. If $A$ is not square, or if $A$ is square but not invertible, then Theorem 1.6.2 does not apply. In these cases $\mathbf{b}$ must usually satisfy certain conditions in order for $A\mathbf{x} = \mathbf{b}$ to be consistent. The following example illustrates how the methods of Section 1.2 can be used to determine such conditions.

### ▶ EXAMPLE 3 Determining Consistency by Elimination

What conditions must $b_1$, $b_2$, and $b_3$ satisfy in order for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 \phantom{{}+x_2} + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

to be consistent?

$$\begin{array}{rr} 1 & -1 \\ 2 & 0 \\ \hline 3 & -1 \end{array}$$

***Solution*** The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \quad 3 \times 4$$

which can be reduced to row echelon form as follows:

$-1R1+R2$
$-2R1+R3$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & -b_1 + b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

← $-1$ times the first row was added to the second and $-2$ times the first row was added to the third.

$-R2$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

← The second row was multiplied by $-1$.

$1R2+R3$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

← The second row was added to the third.

$0 = b_3 - b_2 - b_1$

It is now evident from the third row in the matrix that the system has a solution if and only if $b_1$, $b_2$, and $b_3$ satisfy the condition

$$b_3 - b_2 - b_1 = 0 \quad \text{or} \quad b_3 = b_1 + b_2$$

To express this condition another way, $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a matrix of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}$$

where $b_1$ and $b_2$ are arbitrary.

### ▶ EXAMPLE 4 Determining Consistency by Elimination

What conditions must $b_1$, $b_2$, and $b_3$ satisfy in order for the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \phantom{{}+5x_2} + 8x_3 &= b_3 \end{aligned}$$

to be consistent?

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \qquad\quad + 8x_3 &= b_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{array}\right]$$

$$\xrightarrow[-1R1+R3]{-2R1+R2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & -2b_1+b_2 \\ 0 & -2 & 5 & -b_1+b_3 \end{array}\right]$$

$$\xrightarrow{2R2+R3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & -2b_1+b_2 \\ 0 & 0 & -1 & -5b_1+2b_2+b_3 \end{array}\right]$$

$$\xrightarrow{-R3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & -2b_1+b_2 \\ 0 & 0 & 1 & 5b_1-2b_2-b_3 \end{array}\right]$$

$$\xrightarrow[-3R3+R1]{3R3+R2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -14b_1+6b_2+3b_3 \\ 0 & 1 & 0 & 13b_1-5b_2-3b_3 \\ 0 & 0 & 1 & 5b_1-2b_2-b_3 \end{array}\right]$$

$$2(-2b_1+b_2) + (-b_1+b_3)$$
$$= -4b_1+2b_2-b_1+b_3$$
$$= \boxed{-5b_1+2b_2+b_3}$$

$$3(5b_1-2b_2-b_3) + (-2b_1+b_2)$$
$$15b_1-6b_2-3b_3$$
$$-2b_1+b_2$$
$$-15b_1+6b_2+3b_3$$
$$+b_1$$

$$\xrightarrow[-3R3+R1]{3R3+R2}\left[\begin{array}{ccc|c} 1 & 2 & 0 & -14b_1+6b_2+3b_3 \\ 0 & 1 & 0 & \boxed{13b_1-5b_2-3b_3} \\ 0 & 0 & 1 & 5b_1-2b_2-b_3 \end{array}\right]$$

$$\xrightarrow{-2R2+R1}\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -40b_1+16b_2+9b_3 \\ 0 & 1 & 0 & 13b_1-5b_2-3b_3 \\ 0 & 0 & 1 & 5b_1-2b_2-b_3 \end{array}\right] \end{array}$$

$$\left.\begin{array}{l} -26b_1+10b_2+6b_3 \\ -14b_1+6b_2+3b_3 \end{array}\right\} = -40b_1+16b_2+9b_3$$

$$\left.\begin{array}{l} \underline{x_1} = -40b_1+16b_2+9b_3 \\ \underline{x_2} = 13b_1-5b_2-3b_3 \\ \underline{x_3} = 5b_1-2b_2-b_3 \end{array}\right\}$$

***Solution*** The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{bmatrix}$$

Reducing this to reduced row echelon form yields (verify)

$$\begin{bmatrix} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{bmatrix} \tag{2}$$

In this case there are no restrictions on $b_1$, $b_2$, and $b_3$, so the system has the unique solution

$$x_1 = -40b_1 + 16b_2 + 9b_3, \quad x_2 = 13b_1 - 5b_2 - 3b_3, \quad x_3 = 5b_1 - 2b_2 - b_3 \tag{3}$$

for all values of $b_1$, $b_2$, and $b_3$. ◀

What does the result in Example 4 tell you about the coefficient matrix of the system?

## Exercise Set 1.6

▶ In Exercises 1–8, solve the system by inverting the coefficient matrix and using Theorem 1.6.2. ◀

**1.** $\begin{aligned} x_1 + x_2 &= 2 \\ 5x_1 + 6x_2 &= 9 \end{aligned}$

**2.** $\begin{aligned} 4x_1 - 3x_2 &= -3 \\ 2x_1 - 5x_2 &= 9 \end{aligned}$

**3.** $\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$

**4.** $\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{aligned}$

**5.** $\begin{aligned} x + y + z &= 5 \\ x + y - 4z &= 10 \\ -4x + y + z &= 0 \end{aligned}$

**6.** $\begin{aligned} - x - 2y - 3z &= 0 \\ w + x + 4y + 4z &= 7 \\ w + 3x + 7y + 9z &= 4 \\ -w - 2x - 4y - 6z &= 6 \end{aligned}$

**7.** $\begin{aligned} 3x_1 + 5x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$

**8.** $\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 5x_3 &= b_2 \\ 3x_1 + 5x_2 + 8x_3 &= b_3 \end{aligned}$

▶ In Exercises 9–12, solve the linear systems together by reducing the appropriate augmented matrix. ◀

**9.** $\begin{aligned} x_1 - 5x_2 &= b_1 \\ 3x_1 + 2x_2 &= b_2 \end{aligned}$

(i) $b_1 = 1, \quad b_2 = 4$ (ii) $b_1 = -2, \quad b_2 = 5$

**10.** $\begin{aligned} -x_1 + 4x_2 + x_3 &= b_1 \\ x_1 + 9x_2 - 2x_3 &= b_2 \\ 6x_1 + 4x_2 - 8x_3 &= b_3 \end{aligned}$

(i) $b_1 = 0, \quad b_2 = 1, \quad b_3 = 0$
(ii) $b_1 = -3, \quad b_2 = 4, \quad b_3 = -5$

**11.** $\begin{aligned} 4x_1 - 7x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$

(i) $b_1 = 0, \quad b_2 = 1$ (ii) $b_1 = -4, \quad b_2 = 6$
(iii) $b_1 = -1, \quad b_2 = 3$ (iv) $b_1 = -5, \quad b_2 = 1$

**12.** $\begin{aligned} x_1 + 3x_2 + 5x_3 &= b_1 \\ -x_1 - 2x_2 \phantom{+ 5x_3} &= b_2 \\ 2x_1 + 5x_2 + 4x_3 &= b_3 \end{aligned}$

(i) $b_1 = 1, \quad b_2 = 0, \quad b_3 = -1$
(ii) $b_1 = 0, \quad b_2 = 1, \quad b_3 = 1$
(iii) $b_1 = -1, \quad b_2 = -1, \quad b_3 = 0$

▶ In Exercises 13–17, determine conditions on the $b_i$'s, if any, in order to guarantee that the linear system is consistent. ◀

**13.** $\begin{aligned} x_1 + 3x_2 &= b_1 \\ -2x_1 + x_2 &= b_2 \end{aligned}$

**14.** $\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$

**15.** $\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$

**16.** $\begin{aligned} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{aligned}$

**17.** $\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$

**18.** Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I)\mathbf{x} = \mathbf{0}$ and use this result to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$.

(b) Solve $A\mathbf{x} = 4\mathbf{x}$.

▶ In Exercises 19–20, solve the matrix equation for $X$. ◀

**19.** $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$

**20.** $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix} X = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix}$

### Working with Proofs

**21.** Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of $n$ linear equations in $n$ unknowns that has only the trivial solution. Prove that if $k$ is any positive integer, then the system $A^k\mathbf{x} = \mathbf{0}$ also has only the trivial solution.

**22.** Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of $n$ linear equations in $n$ unknowns, and let $Q$ be an invertible $n \times n$ matrix. Prove that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

**23.** Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations, and let $\mathbf{x}_1$ be a fixed solution. Prove that every solution to the system can be written in the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$, where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$. Prove also that every matrix of this form is a solution.

**24.** Use part (*a*) of Theorem 1.6.3 to prove part (*b*).

### True-False Exercises

**TF.** In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) It is impossible for a system of linear equations to have exactly two solutions.

(b) If $A$ is a square matrix, and if the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the linear system $A\mathbf{x} = \mathbf{c}$ also must have a unique solution.

(c) If $A$ and $B$ are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

(d) If $A$ and $B$ are row equivalent matrices, then the linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set.

(e) Let $A$ be an $n \times n$ matrix and $S$ is an $n \times n$ invertible matrix. If $\mathbf{x}$ is a solution to the linear system $(S^{-1}AS)\mathbf{x} = \mathbf{b}$, then $S\mathbf{x}$ is a solution to the linear system $A\mathbf{y} = S\mathbf{b}$.

(f) Let $A$ be an $n \times n$ matrix. The linear system $A\mathbf{x} = 4\mathbf{x}$ has a unique solution if and only if $A - 4I$ is an invertible matrix.

(g) Let $A$ and $B$ be $n \times n$ matrices. If $A$ or $B$ (or both) are not invertible, then neither is $AB$.

### Working with Technology

**T1.** Colors in print media, on computer monitors, and on television screens are implemented using what are called "color models". For example, in the RGB model, colors are created by mixing percentages of red (R), green (G), and blue (B), and in the YIQ model (used in TV broadcasting), colors are created by mixing percentages of luminescence (Y) with percentages of a chrominance factor (I) and a chrominance factor (Q). The conversion from the RGB model to the YIQ model is accomplished by the matrix equation

$$\begin{bmatrix} Y \\ I \\ Q \end{bmatrix} = \begin{bmatrix} .299 & .587 & .114 \\ .596 & -.275 & -.321 \\ .212 & -.523 & .311 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

What matrix would you use to convert the YIQ model to the RGB model?

**T2.** Let

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 4 & 5 & 1 \\ 0 & 3 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 11 \\ 5 \\ 3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

Solve the linear systems $A\mathbf{x} = B_1$, $A\mathbf{x} = B_2$, $A\mathbf{x} = B_3$ using the method of Example 2.

# 1.7 Diagonal, Triangular, and Symmetric Matrices

In this section we will discuss matrices that have various special forms. These matrices arise in a wide variety of applications and will play an important role in our subsequent work.

## Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. Here are some examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$