(e) For every matrix A, it is true that $(A^T)^T = A$.

(f) If A and B are square matrices of the same order, then
$$\text{tr}(AB) = \text{tr}(A)\text{tr}(B)$$

(g) If A and B are square matrices of the same order, then
$$(AB)^T = A^T B^T$$

(h) For every square matrix A, it is true that $\text{tr}(A^T) = \text{tr}(A)$.

(i) If A is a 6×4 matrix and B is an $m \times n$ matrix such that $B^T A^T$ is a 2×6 matrix, then $m = 4$ and $n = 2$.

(j) If A is an $n \times n$ matrix and c is a scalar, then $\text{tr}(cA) = c\,\text{tr}(A)$.

(k) If A, B, and C are matrices of the same size such that $A - C = B - C$, then $A = B$.

(l) If A, B, and C are square matrices of the same order such that $AC = BC$, then $A = B$.

(m) If $AB + BA$ is defined, then A and B are square matrices of the same size.

(n) If B has a column of zeros, then so does AB if this product is defined.

(o) If B has a column of zeros, then so does BA if this product is defined.

Working with Technology

T1. (a) Compute the product AB of the matrices in Example 5, and compare your answer to that in the text.

(b) Use your technology utility to extract the columns of A and the rows of B, and then calculate the product AB by a column-row expansion.

T2. Suppose that a manufacturer uses Type I items at \$1.35 each, Type II items at \$2.15 each, and Type III items at \$3.95 each. Suppose also that the accompanying table describes the purchases of those items (in thousands of units) for the first quarter of the year. Write down a matrix product, the computation of which produces a matrix that lists the manufacturer's expenditure in each month of the first quarter. Compute that product.

	Type I	Type II	Type III
Jan.	3.1	4.2	3.5
Feb.	5.1	6.8	0
Mar.	2.2	9.5	4.0
Apr.	1.0	1.0	7.4

1.4 Inverses; Algebraic Properties of Matrices

In this section we will discuss some of the algebraic properties of matrix operations. We will see that many of the basic rules of arithmetic for real numbers hold for matrices, but we will also see that some do not.

Properties of Matrix Addition and Scalar Multiplication

The following theorem lists the basic algebraic properties of the matrix operations.

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$ **[Commutative law for matrix addition]**

(b) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(c) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(d) $A(B + C) = AB + AC$ **[Left distributive law]**

(e) $(B + C)A = BA + CA$ **[Right distributive law]**

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

To prove any of the equalities in this theorem we must show that the matrix on the left side has the same size as that on the right and that the corresponding entries on the two sides are the same. Most of the proofs follow the same pattern, so we will prove part (d) as a sample. The proof of the associative law for multiplication is more complicated than the rest and is outlined in the exercises.

Proof (d) We must show that $A(B+C)$ and $AB+AC$ have the same size and that corresponding entries are equal. To form $A(B+C)$, the matrices B and C must have the same size, say $m \times n$, and the matrix A must then have m columns, so its size must be of the form $r \times m$. This makes $A(B+C)$ an $r \times n$ matrix. It follows that $AB+AC$ is also an $r \times n$ matrix and, consequently, $A(B+C)$ and $AB+AC$ have the same size.

Suppose that $A=[a_{ij}]$, $B=[b_{ij}]$, and $C=[c_{ij}]$. We want to show that corresponding entries of $A(B+C)$ and $AB+AC$ are equal; that is,

$$\big(A(B+C)\big)_{ij} = (AB+AC)_{ij}$$

for all values of i and j. But from the definitions of matrix addition and matrix multiplication, we have

$$\begin{aligned}\big(A(B+C)\big)_{ij} &= a_{i1}(b_{1j}+c_{1j}) + a_{i2}(b_{2j}+c_{2j}) + \cdots + a_{im}(b_{mj}+c_{mj}) \\ &= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{im}c_{mj}) \\ &= (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij} \quad ◀\end{aligned}$$

There are three basic ways to prove that two matrices of the same size are equal—prove that corresponding entries are the same, prove that corresponding row vectors are the same, or prove that corresponding column vectors are the same.

Remark Although the operations of matrix addition and matrix multiplication were defined for pairs of matrices, associative laws (b) and (c) enable us to denote sums and products of three matrices as $A+B+C$ and ABC without inserting any parentheses. This is justified by the fact that no matter how parentheses are inserted, the associative laws guarantee that the same end result will be obtained. In general, *given any sum or any product of matrices, pairs of parentheses can be inserted or deleted anywhere within the expression without affecting the end result.*

L.H.S R.H.S

▶ EXAMPLE 1 Associativity of Matrix Multiplication $(AB)C = A(BC)$

As an illustration of the associative law for matrix multiplication, consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

3X2 2X2 2X2

R.H.S

Then

L.H.S

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

so $(AB)C = A(BC)$, as guaranteed by Theorem 1.4.1(c). ◀

Properties of Matrix Multiplication

Do not let Theorem 1.4.1 lull you into believing that *all* laws of real arithmetic carry over to matrix arithmetic. For example, you know that in real arithmetic it is always true that $ab = ba$, which is called the *commutative law for multiplication*. In matrix arithmetic, however, the equality of AB and BA can fail for three possible reasons:

1. AB may be defined and BA may not (for example, if A is 2×3 and B is 3×4).
2. AB and BA may both be defined, but they may have different sizes (for example, if A is 2×3 and B is 3×2). $AB_{2\times 2}$ $BA_{3\times 3}$
3. AB and BA may both be defined and have the same size, but the two products may be different (as illustrated in the next example).

EXAMPLE 2 Order Matters in Matrix Multiplication

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

2x2 2x2

Thus, $AB \neq BA$. ◀

Do not read too much into Example 2—it does not rule out the possibility that AB and BA may be equal in *certain* cases, just that they are not equal in *all* cases. If it so happens that $AB = BA$, then we say that AB and BA ***commute***.

Zero Matrices

A matrix whose entries are all zero is called a ***zero matrix***. Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

2x2 3x3 2x4 4x1 1x1

We will denote a zero matrix by 0 unless it is important to specify its size, in which case we will denote the $m \times n$ zero matrix by $0_{m \times n}$.

It should be evident that if A and 0 are matrices with the same size, then

$$A + 0 = 0 + A = A$$

Thus, 0 plays the same role in this matrix equation that the number 0 plays in the numerical equation $a + 0 = 0 + a = a$.

The following theorem lists the basic properties of zero matrices. Since the results should be self-evident, we will omit the formal proofs.

THEOREM 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

(a) $A + 0 = 0 + A = A$

(b) $A - 0 = A$ $0 - A = -A$

(c) $A - A = A + (-A) = 0$

(d) $0A = 0$

(e) *If $cA = 0$, then $c = 0$ or $A = 0$.*

Since we know that the commutative law of real arithmetic is not valid in matrix arithmetic, it should not be surprising that there are other rules that fail as well. For example, consider the following two laws of real arithmetic:

- If $ab = ac$ and $a \neq 0$, then $b = c$. **[The cancellation law]**
- If $ab = 0$, then at least one of the factors on the left is 0.

The next two examples show that these laws are not true in matrix arithmetic.

▶ EXAMPLE 3 Failure of the Cancellation Law

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

$AB = AC$

$B = C$

$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

We leave it for you to confirm that

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Although $A \neq 0$, canceling A from both sides of the equation $AB = AC$ would lead to the incorrect conclusion that $B = C$. Thus, the cancellation law does not hold, in general, for matrix multiplication (though there may be particular cases where it is true).

▶ EXAMPLE 4 A Zero Product with Nonzero Factors

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} \blacktriangleleft$$

$A \neq 0$

$B \neq 0$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Identity Matrices

A square matrix with 1's on the main diagonal and zeros elsewhere is called an ***identity matrix***. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1x1 2x2 3x3 4x4 I_n

An identity matrix is denoted by the letter I. If it is important to emphasize the size, we will write I_n for the $n \times n$ identity matrix.

To explain the role of identity matrices in matrix arithmetic, let us consider the effect of multiplying a general 2×3 matrix A on each side by an identity matrix. Multiplying on the right by the 3×3 identity matrix yields

$$AI_3 = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and multiplying on the left by the 2×2 identity matrix yields

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

2x2 2x3

$I_3 = 3\times 3$

The same result holds in general; that is, if A is any $m \times n$ matrix, then

$$AI_n = A \quad \text{and} \quad I_mA = A$$

Thus, the identity matrices play the same role in matrix arithmetic that the number 1 plays in the numerical equation $a \cdot 1 = 1 \cdot a = a$.

As the next theorem shows, identity matrices arise naturally in studying reduced row echelon forms of *square* matrices.

$A_{n \times n}$

R reduced row echelon form

THEOREM 1.4.3 *If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n.*

Proof Suppose that the reduced row echelon form of A is

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

Either the last row in this matrix consists entirely of zeros or it does not. If not, the matrix contains no zero rows, and consequently each of the n rows has a leading entry of 1. Since these leading 1's occur progressively farther to the right as we move down the matrix, each of these 1's must occur on the main diagonal. Since the other entries in the same column as one of these 1's are zero, R must be I_n. Thus, either R has a row of zeros or $R = I_n$. ◀

$2 \cdot \frac{1}{2}$

2^{-1}

Inverse of a Matrix

In real arithmetic every nonzero number a has a reciprocal $a^{-1} (= 1/a)$ with the property

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

The number a^{-1} is sometimes called the *multiplicative inverse* of a. Our next objective is to develop an analog of this result for matrix arithmetic. For this purpose we make the following definition.

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

$\exists B : AB = BA = I$

$\nexists B \Rightarrow A$ singular.

Remark The relationship $AB = BA = I$ is not changed by interchanging A and B, so if A is invertible and B is an inverse of A, then it is also true that B is invertible, and A is an inverse of B. Thus, when

$$AB = BA = I$$

we say that A and B are *inverses of one another*.

A : invertible
B : invertible

▶ EXAMPLE 5 An Invertible Matrix

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $\qquad$ $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A^{-1} \neq \frac{1}{A}$

$AA^{-1} = A^{-1}A = I$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

EXAMPLE 6 A Class of Singular Matrices

A square matrix with a row or column of zeros is singular. To help understand why this is so, consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

$c_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $c_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$

To prove that A is singular we must show that there is no 3×3 matrix B such that $AB = BA = I$. For this purpose let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{0}$ be the column vectors of A. Thus, for any 3×3 matrix B we can express the product BA as

$$BA = B[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}] = [B\mathbf{c}_1 \quad B\mathbf{c}_2 \quad \mathbf{0}] \quad \textbf{[Formula (6) of Section 1.3]}$$

The column of zeros shows that $BA \neq I$ and hence that A is singular. ◀

As in Example 6, we will frequently denote a zero matrix with one row or one column by a boldface zero.

B & C inverse of A

$BA = AB = I \wedge CA = AC = I$

Properties of Inverses

It is reasonable to ask whether an invertible matrix can have more than one inverse. The next theorem shows that the answer is no—*an invertible matrix has exactly one inverse.*

$p \rightarrow q$

THEOREM 1.4.4 *If B and C are both inverses of the matrix A, then $B = C$.*

$BA = I$

$(BA)C = I.C$

$B(AC) = I.C$

$B.I = I.C$

$B = C$

Proof Since B is an inverse of A, we have $BA = I$. Multiplying both sides on the right by C gives $(BA)C = IC = C$. But it is also true that $(BA)C = B(AC) = BI = B$, so $C = B$. ◀

As a consequence of this important result, we can now speak of "the" inverse of an invertible matrix. If A is invertible, then its inverse will be denoted by the symbol A^{-1}. Thus,

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I \tag{1}$$

WARNING The symbol A^{-1} should not be interpreted as $1/A$. Division by matrices will not be a defined operation in this text.

The inverse of A plays much the same role in matrix arithmetic that the reciprocal a^{-1} plays in the numerical relationships $aa^{-1} = 1$ and $a^{-1}a = 1$.

In the next section we will develop a method for computing the inverse of an invertible matrix of any size. For now we give the following theorem that specifies conditions under which a 2×2 matrix is invertible and provides a simple formula for its inverse.

Historical Note The formula for A^{-1} given in Theorem 1.4.5 first appeared (in a more general form) in Arthur Cayley's 1858 *Memoir on the Theory of Matrices*. The more general result that Cayley discovered will be studied later.

The quantity $ad - bc$ in Theorem 1.4.5 is called the ***determinant*** of the 2×2 matrix A and is denoted by

$$\det(A) = ad - bc$$

or alternatively by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \underbrace{\frac{1}{ad - bc}}_{\text{scalar}} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{matrix}} \tag{2}$$

We will omit the proof, because we will study a more general version of this theorem later. For now, you should at least confirm the validity of Formula (2) by showing that $AA^{-1} = A^{-1}A = I$.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

▲ Figure 1.4.1

Remark Figure 1.4.1 illustrates that the determinant of a 2×2 matrix A is the product of the entries on its main diagonal minus the product of the entries *off* its main diagonal.

▶ EXAMPLE 7 Calculating the Inverse of a 2 × 2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

(a) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

$$A^{-1} = \frac{1}{6-6}\begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix}$$

$\Rightarrow A$ is singular

$$A^{-1} = \frac{1}{(6)(2)-(5)(1)}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{-1}{7} \\ \frac{-5}{7} & \frac{6}{7} \end{bmatrix}$$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

▶ EXAMPLE 8 Solution of a Linear System by Matrix Inversion

A problem that arises in many applications is to solve a pair of equations of the form

$$\begin{aligned} u &= ax + by \\ v &= cx + dy \end{aligned}$$

2 variables
2 equations

for x and y in terms of u and v. One approach is to treat this as a linear system of two equations in the unknowns x and y and use Gauss–Jordan elimination to solve for x and y. However, because the coefficients of the unknowns are *literal* rather than *numerical*, this procedure is a little clumsy. As an alternative approach, let us replace the two equations by the single matrix equation

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$2x = 5 \qquad x = \frac{5}{2}$$

which we can rewrite as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If we assume that the 2×2 matrix is invertible (i.e., $ad - bc \neq 0$), then we can multiply through on the left by the inverse and rewrite the equation as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$I \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$(ab)^2 = a^2 b^2$

$(ab)^{-1} = a^{-1} b^{-1}$

Using Theorem 1.4.5, we can rewrite this equation as

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

from which we obtain

$$x = \frac{du - bv}{ad - bc}, \quad y = \frac{av - cu}{ad - bc} \quad ◀$$

The next theorem is concerned with inverses of matrix products.

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

Proof We can establish the invertibility and obtain the stated formula at the same time by showing that

$$\boxed{(AB)(B^{-1}A^{-1}) \overset{??}{=} (B^{-1}A^{-1})(AB) \overset{??}{=} I}$$

But

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly, $(B^{-1}A^{-1})(AB) = I$. ◀ $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

Although we will not prove it, this result can be extended to three or more factors:

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

▶ EXAMPLE 9 The Inverse of a Product

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

$(AB)^{-1} = \frac{1}{56-54}\begin{bmatrix} 8 & -6 \\ -9 & 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 8 & -6 \\ -9 & 7 \end{bmatrix}$

We leave it for you to show that

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \boxed{\begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}}$$

and also that

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}, \quad B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$ as guaranteed by Theorem 1.4.6. ◀

If a product of matrices is singular, then at least one of the factors must be singular. Why?

$a^2 = a.a$

$a^0 = 1$

Powers of a Matrix If A is a *square* matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad \text{and} \quad A^n = AA \cdots A \quad \textbf{[}n \textbf{ factors]}$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \cdots A^{-1} \quad \textbf{[}n \textbf{ factors]}$$

$a^2 \cdot a^3 = a^5$

$(a^2)^4 = a^8$

Because these definitions parallel those for real numbers, the usual laws of nonnegative exponents hold; for example,

$$A^r A^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}$$

In addition, we have the following properties of negative exponents.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Is A^{100} invertible or not?

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

(*a*) *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*

(*b*) *A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.*

(*c*) *kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.*

$\frac{1}{k}$

$A^{-1} = \frac{1}{3-2}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

We will prove part (*c*) and leave the proofs of parts (*a*) and (*b*) as exercises.

Proof (c) Properties (*m*) and (*l*) of Theorem 1.4.1 imply that

$$(kA)(k^{-1}A^{-1}) = k^{-1}(kA)A^{-1} = (k^{-1}k)AA^{-1} = (1)I = I$$

and similarly, $(k^{-1}A^{-1})(kA) = I$. Thus, kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$. ◀

$(ab)A$

$a(bA)$

▶ EXAMPLE 10 Properties of Exponents

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

so, as expected from Theorem 1.4.7(*b*),

$$(A^3)^{-1} = \frac{1}{(11)(41) - (30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

▶ EXAMPLE 11 The Square of a Matrix Sum

In real arithmetic, where we have a commutative law for multiplication, we can write

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

However, in matrix arithmetic, where we have no commutative law for multiplication, the best we can do is to write

$$(A+B)^2 = A^2 + AB + BA + B^2$$

It is only in the special case where A and B *commute* (i.e., $AB = BA$) that we can go a step further and write

$$(A+B)^2 = A^2 + 2AB + B^2 \; ◀$$

$(A+B)^2 = (A+B)(A+B)$

$= A \cdot A + A \cdot B + B \cdot A + B \cdot B$

$= A^2 + AB + BA + B^2$

$= A^2 + 2AB + B^2$ if A and B commute

$AB = BA$

Matrix Polynomials

If A is a square matrix, say $n \times n$, and if

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$$

$P(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m$

is any polynomial, then we define the $n \times n$ matrix $p(A)$ to be

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m \tag{3}$$

where I is the $n \times n$ identity matrix; that is, $p(A)$ is obtained by substituting A for x and replacing the constant term a_0 by the matrix $a_0 I$. An expression of form (3) is called a ***matrix polynomial in A***.

EXAMPLE 12 A Matrix Polynomial

Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$. ◀

Remark It follows from the fact that $A^r A^s = A^{r+s} = A^{s+r} = A^s A^r$ that powers of a square matrix commute, and since a matrix polynomial in A is built up from powers of A, any two matrix polynomials in A also commute; that is, for any polynomials p_1 and p_2 we have

$$p_1(A)p_2(A) = p_2(A)p_1(A) \tag{4}$$

Properties of the Transpose

The following theorem lists the main properties of the transpose.

$(kA)^{-1} = k^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

THEOREM 1.4.8 *If the sizes of the matrices are such that the stated operations can be performed, then:*

(*a*) $(A^T)^T = A$

(*b*) $(A + B)^T = A^T + B^T$

(*c*) $(A - B)^T = A^T - B^T$

(*d*) $(kA)^T = kA^T$

(*e*) $(AB)^T = B^T A^T$

If you keep in mind that transposing a matrix interchanges its rows and columns, then you should have little trouble visualizing the results in parts (*a*)–(*d*). For example, part (*a*) states the obvious fact that interchanging rows and columns twice leaves a matrix unchanged; and part (*b*) states that adding two matrices and then interchanging the rows and columns produces the same result as interchanging the rows and columns before adding. We will omit the formal proofs. Part (*e*) is less obvious, but for brevity we will omit its proof as well. The result in that part can be extended to three or more factors and restated as:

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

$5^{-1} = \frac{1}{5}$

$5 \cdot \frac{1}{5} = 1$

The following theorem establishes a relationship between the inverse of a matrix and the inverse of its transpose.

THEOREM 1.4.9 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

Proof We can establish the invertibility and obtain the formula at the same time by showing that

$$A^T(A^{-1})^T \overset{??}{=} (A^{-1})^T A^T \overset{??}{=} I$$

But from part (*e*) of Theorem 1.4.8 and the fact that $I^T = I$, we have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

which completes the proof. ◀

▶ EXAMPLE 13 Inverse of a Transpose

Consider a general 2×2 invertible matrix and its transpose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$A^{-1} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{d}{ad-cb} & \frac{-b}{ad-cb} \\ \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}$

$(A^{-1})^T = \begin{bmatrix} \frac{d}{ad-cb} & \frac{-c}{ad-cb} \\ \frac{-b}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}$

Since A is invertible, its determinant $ad - bc$ is nonzero. But the determinant of A^T is also $ad - bc$ (verify), so A^T is also invertible. It follows from Theorem 1.4.5 that

$$(A^T)^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{c}{ad-bc} \\ -\dfrac{b}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

which is the same matrix that results if A^{-1} is transposed (verify). Thus,

$$(A^T)^{-1} = (A^{-1})^T$$

as guaranteed by Theorem 1.4.9. ◀

Exercise Set 1.4

▶ In Exercises 1–2, verify that the following matrices and scalars satisfy the stated properties of Theorem 1.4.1.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}, \quad a = 4, \quad b = -7 \;◀$$

1. (a) The associative law for matrix addition.

(b) The associative law for matrix multiplication.

(c) The left distributive law.

(d) $(a + b)C = aC + bC$

2. (a) $a(BC) = (aB)C = B(aC)$

(b) $A(B - C) = AB - AC$ (c) $(B + C)A = BA + CA$

(d) $a(bC) = (ab)C$

▶ In Exercises 3–4, verify that the matrices and scalars in Exercise 1 satisfy the stated properties. ◀

3. (a) $(A^T)^T = A$ (b) $(AB)^T = B^T A^T$

4. (a) $(A + B)^T = A^T + B^T$ (b) $(aC)^T = aC^T$

▶ In Exercises 5–8, use Theorem 1.4.5 to compute the inverse of the matrix. ◀

5. $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$ **6.** $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

7. $C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

8. $D = \begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$

9. Find the inverse of

$$\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

10. Find the inverse of

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

▶ In Exercises 11–14, verify that the equations are valid for the matrices in Exercises 5–8. ◀

11. $(A^T)^{-1} = (A^{-1})^T$

12. $(A^{-1})^{-1} = A$

13. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

14. $(ABC)^T = C^TB^TA^T$

▶ In Exercises 15–18, use the given information to find A. ◀

15. $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$

16. $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

17. $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

18. $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$

▶ In Exercises 19–20, compute the following using the given matrix A.

(a) A^3 (b) A^{-3} (c) $A^2 - 2A + I$ ◀

19. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

20. $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

▶ In Exercises 21–22, compute $p(A)$ for the given matrix A and the following polynomials.

(a) $p(x) = x - 2$

(b) $p(x) = 2x^2 - x + 1$

(c) $p(x) = x^3 - 2x + 1$ ◀

21. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

22. $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

▶ In Exercises 23–24, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad ◀$$

23. Find all values of a, b, c, and d (if any) for which the matrices A and B commute.

24. Find all values of a, b, c, and d (if any) for which the matrices A and C commute.

▶ In Exercises 25–28, use the method of Example 8 to find the unique solution of the given linear system. ◀

25. $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \end{aligned}$

26. $\begin{aligned} -x_1 + 5x_2 &= 4 \\ -x_1 - 3x_2 &= 1 \end{aligned}$

27. $\begin{aligned} 6x_1 + x_2 &= 0 \\ 4x_1 - 3x_2 &= -2 \end{aligned}$

28. $\begin{aligned} 2x_1 - 2x_2 &= 4 \\ x_1 + 4x_2 &= 4 \end{aligned}$

▶ If a polynomial $p(x)$ can be factored as a product of lower degree polynomials, say

$$p(x) = p_1(x)p_2(x)$$

and if A is a square matrix, then it can be proved that

$$p(A) = p_1(A)p_2(A)$$

In Exercises 29–30, verify this statement for the stated matrix A and polynomials

$$p(x) = x^2 - 9, \quad p_1(x) = x + 3, \quad p_2(x) = x - 3 \quad ◀$$

29. The matrix A in Exercise 21.

30. An arbitrary square matrix A.

31. (a) Give an example of two 2×2 matrices such that

$$(A + B)(A - B) \neq A^2 - B^2$$

(b) State a valid formula for multiplying out

$$(A + B)(A - B)$$

(c) What condition can you impose on A and B that will allow you to write $(A + B)(A - B) = A^2 - B^2$?

32. The numerical equation $a^2 = 1$ has exactly two solutions. Find at least eight solutions of the matrix equation $A^2 = I_3$. [*Hint:* Look for solutions in which all entries off the main diagonal are zero.]

33. (a) Show that if a square matrix A satisfies the equation $A^2 + 2A + I = 0$, then A must be invertible. What is the inverse?

(b) Show that if $p(x)$ is a polynomial with a nonzero constant term, and if A is a square matrix for which $p(A) = 0$, then A is invertible.

34. Is it possible for A^3 to be an identity matrix without A being invertible? Explain.

35. Can a matrix with a row of zeros or a column of zeros have an inverse? Explain.

36. Can a matrix with two identical rows or two identical columns have an inverse? Explain.

▶ In Exercises 37–38, determine whether A is invertible, and if so, find the inverse. [*Hint:* Solve $AX = I$ for X by equating corresponding entries on the two sides.] ◀

37. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **38.** $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

▶ In Exercises 39–40, simplify the expression assuming that A, B, C, and D are invertible. ◀

39. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$

40. $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

41. Show that if R is a $1 \times n$ matrix and C is an $n \times 1$ matrix, then $RC = \text{tr}(CR)$.

42. If A is a square matrix and n is a positive integer, is it true that $(A^n)^T = (A^T)^n$? Justify your answer.

43. (a) Show that if A is invertible and $AB = AC$, then $B = C$.

(b) Explain why part (a) and Example 3 do not contradict one another.

44. Show that if A is invertible and k is any nonzero scalar, then $(kA)^n = k^n A^n$ for all integer values of n.

45. (a) Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

(b) What does the result in part (a) tell you about the matrix $A^{-1} + B^{-1}$?

46. A square matrix A is said to be ***idempotent*** if $A^2 = A$.

(a) Show that if A is idempotent, then so is $I - A$.

(b) Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

47. Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$$

48. Show that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfies the equation

$$A^2 - (a + d)A + (ad - bc)I = 0$$

49. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} = C^T$$

50. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$ABC^T DBA^T C = AB^T$$

Working with Proofs

▶ In Exercises 51–58, prove the stated result. ◀

51. Theorem 1.4.1(*a*) **52.** Theorem 1.4.1(*b*)

53. Theorem 1.4.1(*f*) **54.** Theorem 1.4.1(*c*)

55. Theorem 1.4.2(*c*) **56.** Theorem 1.4.2(*b*)

57. Theorem 1.4.8(*d*) **58.** Theorem 1.4.8(*e*)

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two $n \times n$ matrices, A and B, are inverses of one another if and only if $AB = BA = 0$.

(b) For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$.

(c) For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$.

(d) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(e) If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^T B^T$.

(f) The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

(g) If A and B are matrices of the same size and k is a constant, then $(kA + B)^T = kA^T + B^T$.

(h) If A is an invertible matrix, then so is A^T.

(i) If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ and I is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots + a_m$.

(j) A square matrix containing a row or column of zeros cannot be invertible.

(k) The sum of two invertible matrices of the same size must be invertible.