41. Describe all possible reduced row echelon forms of

(a) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ (b) $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix}$

42. Consider the system of equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \\ ex + fy &= 0 \end{aligned}$$

Discuss the relative positions of the lines $ax + by = 0$, $cx + dy = 0$, and $ex + fy = 0$ when the system has only the trivial solution and when it has nontrivial solutions.

Working with Proofs

43. (a) Prove that if $ad - bc \neq 0$, then the reduced row echelon form of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Use the result in part (a) to prove that if $ad - bc \neq 0$, then the linear system

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \end{aligned}$$

has exactly one solution.

True-False Exercises

TF. In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) If a matrix is in reduced row echelon form, then it is also in row echelon form.

(b) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form.

(c) Every matrix has a unique row echelon form.

(d) A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.

(e) All leading 1's in a matrix in row echelon form must occur in different columns.

(f) If every column of a matrix in row echelon form has a leading 1, then all entries that are not leading 1's are zero.

(g) If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.

(h) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

(i) If a linear system has more unknowns than equations, then it must have infinitely many solutions.

Working with Technology

T1. Find the reduced row echelon form of the augmented matrix for the linear system:

$$\begin{array}{rcrcrcrcr} 6x_1 & + & x_2 & & & + & 4x_4 & = & -3 \\ -9x_1 & + & 2x_2 & + & 3x_3 & - & 8x_4 & = & 1 \\ 7x_1 & & & - & 4x_3 & + & 5x_4 & = & 2 \end{array}$$

Use your result to determine whether the system is consistent and, if so, find its solution.

T2. Find values of the constants A, B, C, and D that make the following equation an identity (i.e., true for all values of x).

$$\frac{3x^3 + 4x^2 - 6x}{(x^2 + 2x + 2)(x^2 - 1)} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{C}{x - 1} + \frac{D}{x + 1}$$

[*Hint:* Obtain a common denominator on the right, and then equate corresponding coefficients of the various powers of x in the two numerators. Students of calculus will recognize this as a problem in partial fractions.]

1.3 Matrices and Matrix Operations

Rectangular arrays of real numbers arise in contexts other than as augmented matrices for linear systems. In this section we will begin to study matrices as objects in their own right by defining operations of addition, subtraction, and multiplication on them.

Matrix Notation and Terminology

In Section 1.2 we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate systems of linear equations. However, rectangular arrays of numbers occur in other contexts as well. For example, the following rectangular array with three rows and seven columns might describe the number of hours that a student spent studying three subjects during a certain week:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Math	2	3	2	4	1	4	2
History	0	3	1	4	3	2	2
Language	4	1	3	1	0	0	2

If we suppress the headings, then we are left with the following rectangular array of numbers with three rows and seven columns, called a "matrix":

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 4 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 & 2 \\ 4 & 1 & 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

More generally, we make the following definition.

DEFINITION 1 A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix.

Matrix brackets are often omitted from 1×1 matrices, making it impossible to tell, for example, whether the symbol 4 denotes the number "four" or the matrix [4]. This rarely causes problems because it is usually possible to tell which is meant from the context.

A, B, C, D

E, F, G

▶ EXAMPLE 1 Examples of Matrices

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4] \quad ◀$$

3×2 1×4 3×3 2×1 1×1

The ***size*** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. For example, the first matrix in Example 1 has three rows and two columns, so its size is 3 by 2 (written 3×2). In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices in Example 1 have sizes 1×4, 3×3, 2×1, and 1×1, respectively.

A matrix with only one row, such as the second in Example 1, is called a ***row vector*** (or a ***row matrix***), and a matrix with only one column, such as the fourth in that example, is called a ***column vector*** (or a ***column matrix***). The fifth matrix in that example is both a row vector and a column vector.

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities; thus we might write

entries → $a_{22} = 4$

$a_{13} = 7$

row → a_{ij} ← column

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & \boxed{4} & 2 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

c_{ij}

When discussing matrices, it is common to refer to numerical quantities as ***scalars***. Unless stated otherwise, *scalars will be real numbers*; complex scalars will be considered later in the text.

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij}. Thus a general 3×4 matrix might be written as

$[a_{ij}] = A$

matrix →

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times 4}$$

and a general $m \times n$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \tag{1}$$

When a compact notation is desired, the preceding matrix can be written as

$$[a_{ij}]_{m\times n} \quad \text{or} \quad [a_{ij}]$$

the first notation being used when it is important in the discussion to know the size, and the second when the size need not be emphasized. Usually, we will match the letter denoting a matrix with the letter denoting its entries; thus, for a matrix B we would generally use b_{ij} for the entry in row i and column j, and for a matrix C we would use the notation c_{ij}.

The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix (1) above, we have

$$(A)_{ij} = a_{ij}$$

$(A)_{\underset{\uparrow\ \uparrow}{22}} = 0$

and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2$, $(A)_{12} = -3$, $(A)_{21} = 7$, and $(A)_{22} = 0$.

Row and column vectors are of special importance, and it is common practice to denote them by boldface lowercase letters rather than capital letters. For such matrices, double subscripting of the entries is unnecessary. Thus a general $1 \times n$ row vector $\mathbf{a}$ and a general $m \times 1$ column vector $\mathbf{b}$ would be written as

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$A_{m\times n}$

if $m=n$

$A_{5\times 5}$

$A_{4\times 4}$

A matrix with n rows and n columns is said to be a ***square matrix of order n***.

A matrix A with n rows and n columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (2) are said to be on the ***main diagonal*** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

Operations on Matrices

So far, we have used matrices to abbreviate the work in solving systems of linear equations. For other applications, however, it is desirable to develop an "arithmetic of matrices" in which matrices can be added, subtracted, and multiplied in a useful way. The remainder of this section will be devoted to developing this arithmetic.

DEFINITION 2 Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

$A = B$

① size

② entries # $\boxed{=}$

$A = B$ if $\boxed{x=5}$

EXAMPLE 2 Equality of Matrices

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

2x2 2x2

If $x = 5$, then $A = B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are equal. There is no value of x for which $A = C$ since A and C have different sizes. ◀

The equality of two matrices

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

of the same size can be expressed either by writing

$$(A)_{ij} = (B)_{ij}$$

or by writing

$$a_{ij} = b_{ij}$$

where it is understood that the equalities hold for all values of i and j.

DEFINITION 3 If A and B are matrices of the same size, then the ***sum*** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference*** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

In matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad (A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

EXAMPLE 3 Addition and Subtraction

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

3x4 3x4 2x2

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined. ◀

DEFINITION 4 If A is any matrix and c is any scalar, then the ***product*** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a ***scalar multiple*** of A.

In matrix notation, if $A = [a_{ij}]$, then

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

EXAMPLE 4 Scalar Multiples

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \tfrac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$. ◀

Thus far we have defined multiplication of a matrix by a scalar but not the multiplication of two matrices. Since matrices are added by adding corresponding entries and subtracted by subtracting corresponding entries, it would seem natural to define multiplication of matrices by multiplying corresponding entries. However, it turns out that such a definition would not be very useful for most problems. Experience has led mathematicians to the following more useful definition of matrix multiplication.

$A \quad B$

$m \times r \quad r \times n$

$(AB)_{m\times n}$

DEFINITION 5 If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the ***product*** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

$A_{3\times5} \quad B_{5\times1} = AB_{3\times1}$

$A_{3\times2} \quad B_{3\times2} = AB$ undefined.

$2 \neq 3$

$(AB)_{24} = 2\times3+6\times1+0\times2$

$= 6+6+0$

$= 12$

EXAMPLE 5 Multiplying Matrices

Consider the matrices

$$A = \begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}, \quad B = \begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix} = \begin{bmatrix}\square & \square & \square & \square\\ \square & \square & 26 & \square\end{bmatrix}$$

$$(2\cdot4)+(6\cdot3)+(0\cdot5)=26$$

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix} \quad \begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

$$= \begin{bmatrix}4+0+8 & 1-2+28 & 4+6+20 & 3+2+8\\ 8+0+0 & 2-6+0 & 8+18+0 & 6+6+0\end{bmatrix}$$

$$= \begin{bmatrix}12 & +27 & 30 & 13\\ 8 & -4 & 26 & 12\end{bmatrix}$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix} = \begin{bmatrix}\square & \square & \square & 13\\ \square & \square & \square & \square\end{bmatrix}$$

$$(1\cdot3)+(2\cdot1)+(4\cdot2)=13$$

The computations for the remaining entries are

$$\begin{aligned}
(1\cdot4)+(2\cdot0)+(4\cdot2) &= 12\\
(1\cdot1)-(2\cdot1)+(4\cdot7) &= 27\\
(1\cdot4)+(2\cdot3)+(4\cdot5) &= 30\\
(2\cdot4)+(6\cdot0)+(0\cdot2) &= 8\\
(2\cdot1)-(6\cdot1)+(0\cdot7) &= -4\\
(2\cdot3)+(6\cdot1)+(0\cdot2) &= 12
\end{aligned}
\qquad
AB = \begin{bmatrix}12 & 27 & 30 & 13\\ 8 & -4 & 26 & 12\end{bmatrix}$$

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB. If this condition is not satisfied, the product is undefined. A convenient way to determine whether a product of two matrices is defined is to write down the size of the first factor and, to the right of it, write down the size of the second factor. If, as in (3), the inside numbers are the same, then the product is defined. The outside numbers then give the size of the product.

$$\begin{array}{ccccccc} \multicolumn{3}{c}{A} & \multicolumn{3}{c}{B} & AB \\ m & \times & r & r & \times & n = & m \times n \\ & & \multicolumn{2}{c}{\text{Inside}} & & & \\ \multicolumn{6}{c}{\text{Outside}} & \end{array} \tag{3}$$

EXAMPLE 6 Determining Whether a Product Is Defined

Suppose that A, B, and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3 \end{array}$$

AB = 3x7
AC = undefined
BA = undefined CA = 7x3
BC = 4x3

Then by (3), AB is defined and is a 3×7 matrix; BC is defined and is a 4×3 matrix; and CA is defined and is a 7×4 matrix. The products AC, CB, and BA are all undefined. ◀

In general, if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then, as illustrated by the shading in the following display,

A

A mxr

B rxn

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \tag{4}$$

mxn

(AB) 56

the entry $(AB)_{ij}$ in row i and column j of AB is given by

i

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \tag{5}$$

Formula (5) is called the ***row-column rule*** for matrix multiplication.

Partitioned Matrices

A matrix can be subdivided or ***partitioned*** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For example, the following are three possible partitions of a general 3×4 matrix A—the first is a partition of A into


Gotthold Eisenstein (1823–1852)

Historical Note The concept of matrix multiplication is due to the German mathematician Gotthold Eisenstein, who introduced the idea around 1844 to simplify the process of making substitutions in linear systems. The idea was then expanded on and formalized by Cayley in his *Memoir on the Theory of Matrices* that was published in 1858. Eisenstein was a pupil of Gauss, who ranked him as the equal of Isaac Newton and Archimedes. However, Eisenstein, suffering from bad health his entire life, died at age 30, so his potential was never realized.

[*Image: http://www-history.mcs.st-andrews.ac.uk/Biographies/Eisenstein.html*] four ***submatrices*** A_{11}, A_{12}, A_{21}, and A_{22}; the second is a partition of A into its row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$; and the third is a partition of A into its column vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_4$:

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

$r_1 = [a_{11} \; a_{12} \; a_{13} \; a_{14}]$

$r_2 = [a_{21} \; a_{22} \; a_{23} \; a_{24}]$

$r_3 = [a_{31} \; a_{32} \; a_{33} \; a_{34}]$ rows $\rightarrow$

$c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, c_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$ column $\rightarrow$

$c_4 = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$

$A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$A_{12} = \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix}$

$A_{21} = [a_{31} \; a_{32} \; a_{33}]$

$A_{22} = [a_{34}]$

Matrix Multiplication by Columns and by Rows

Partitioning has many uses, one of which is for finding particular rows or columns of a matrix product AB without computing the entire product. Specifically, the following formulas, whose proofs are left as exercises, show how individual column vectors of AB can be obtained by partitioning B into column vectors and how individual row vectors of AB can be obtained by partitioning A into row vectors.

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n] \tag{6}$$

(*AB* computed column by column)

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix} \tag{7}$$

(*AB* computed row by row)

In words, these formulas state that

$$j\text{th column vector of } AB = A[j\text{th column vector of } B] \tag{8}$$

$$i\text{th row vector of } AB = [i\text{th row vector of } A]B \tag{9}$$

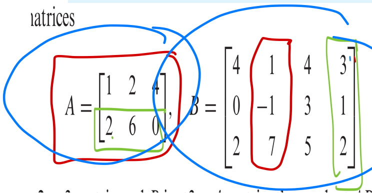
We now have three methods for computing a product of two matrices, entry by entry using Definition 5, column by column using Formula (8), and row by row using Formula (9). We will call these the ***entry method***, the ***row method***, and the ***column method***, respectively.

▶ EXAMPLE 7 Example 5 Revisited

If A and B are the matrices in Example 5, then from (8) the second column vector of AB can be obtained by the computation

$$\overset{A}{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

Second column of B ↑ Second column of AB ↑


ıatrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

second column vector of AB

$= \underset{2\times3}{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$ 2×1

first row vector of AB

$= \underset{1\times3}{[1 \; 2 \; 4]} \underset{3\times4}{\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}} = [12 \; 27 \; 30 \; 13]$

$1\times4 + 2\times0 + 4\times2$ $\quad 1\times1 - 2 + 28$

and from (9) the first row vector of AB can be obtained by the computation

$$\underset{\text{First row of } A}{[1\quad 2\quad 4]}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix} = \underset{\text{First row of } AB}{[12\quad 27\quad 30\quad 13]}$$

Matrix Products as Linear Combinations

The following definition provides yet another way of thinking about matrix multiplication.

DEFINITION 6 If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_rA_r$$

is called a ***linear combination*** of $A_1, A_2, \ldots, A_r$ with ***coefficients*** $c_1, c_2, \ldots, c_r$.

Definition 6 is applicable, in particular, to row and column vectors. Thus, for example, a *linear combination* of column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ of the same size is an expression of the form

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_r\mathbf{x}_r$$

To see how matrix products can be viewed as linear combinations, let A be an $m \times n$ matrix and $\mathbf{x}$ an $n \times 1$ column vector, say

$$A = \underset{m\times n}{\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}} \quad\text{and}\quad \mathbf{x} = \underset{n\times 1}{\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}}$$

$(A\,x)$ $m\times 1$

Then

$$A\mathbf{x} = \begin{bmatrix}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n\\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\end{bmatrix} = x_1\begin{bmatrix}a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{bmatrix} + x_2\begin{bmatrix}a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{bmatrix} + \cdots + x_n\begin{bmatrix}a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{bmatrix} \tag{10}$$

This proves the following theorem.

THEOREM 1.3.1 *If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.*

▶ EXAMPLE 8 Matrix Products as Linear Combinations

The matrix product

$$\underset{A}{\overset{3\times 3}{\begin{bmatrix}-1 & 3 & 2\\ 1 & 2 & -3\\ 2 & 1 & -2\end{bmatrix}}}\overset{3\times 1}{\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}} = \overset{3\times 1}{\begin{bmatrix}1\\ -9\\ -3\end{bmatrix}} \longrightarrow \text{ضرب عادي}$$

can be written as the following linear combination of column vectors:

$$2\begin{bmatrix}-1\\ 1\\ 2\end{bmatrix} - 1\begin{bmatrix}3\\ 2\\ 1\end{bmatrix} + 3\begin{bmatrix}2\\ -3\\ -2\end{bmatrix} = \begin{bmatrix}1\\ -9\\ -3\end{bmatrix}$$

$$Ax = 2\begin{bmatrix}-1\\ 1\\ 2\end{bmatrix} - 1\begin{bmatrix}3\\ 2\\ 1\end{bmatrix} + 3\begin{bmatrix}2\\ -3\\ -2\end{bmatrix} = \begin{bmatrix}-2\\ 2\\ 4\end{bmatrix} + \begin{bmatrix}-3\\ -2\\ -1\end{bmatrix} + \begin{bmatrix}6\\ -9\\ -6\end{bmatrix} = \begin{bmatrix}1\\ -9\\ -3\end{bmatrix}$$

EXAMPLE 9 Columns of a Product *AB* as Linear Combinations

We showed in Example 5 that

$$AB = \overset{A}{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix} \rightarrow$$

ضرب عادي

It follows from Formula (6) and Theorem 1.3.1 that the jth column vector of AB can be expressed as a linear combination of the column vectors of A in which the coefficients in the linear combination are the entries from the jth column of B. The computations are as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix} ◀$$

$$4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\cdot\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\cdot\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 6 \end{bmatrix} + (7)\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Column-Row Expansion

Partitioning provides yet another way to view matrix multiplication. Specifically, suppose that an $m \times r$ matrix A is partitioned into its r column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ (each of size $m \times 1$) and an $r \times n$ matrix B is partitioned into its r row vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_r$ (each of size $1 \times n$). Each term in the sum

$m \times 1$

$1 \times n$

$$\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$$

$m \times n + m \times n - \quad = m \times n$

has size $m \times n$ so the sum itself is an $m \times n$ matrix. We leave it as an exercise for you to verify that the entry in row i and column j of the sum is given by the expression on the right side of Formula (5), from which it follows that

$$AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r \tag{11}$$

We call (11) the ***column-row expansion*** of AB.

EXAMPLE 10 Column-Row Expansion

Find the column-row expansion of the product

$$AB = \overset{A}{\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix}} \tag{12}$$

C_1 C_2 r_1 r_2

Solution The column vectors of A and the row vectors of B are, respectively,

$c_1 r_1\ (2\times3)$

$c_2 r_2\ (2\times3)$

$$\mathbf{c}_1 = \underset{2\times1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \quad \mathbf{c}_2 = \underset{2\times1}{\begin{bmatrix} 3 \\ -1 \end{bmatrix}}; \quad \mathbf{r}_1 = \underset{1\times3}{\begin{bmatrix} 2 & 0 & 4 \end{bmatrix}}, \quad \mathbf{r}_2 = \underset{1\times3}{\begin{bmatrix} -3 & 5 & 1 \end{bmatrix}}$$

Thus, it follows from (11) that the column-row expansion of AB is

$$
\begin{aligned}
AB &= \overset{c_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \overset{r_1}{\begin{bmatrix} 2 & 0 & 4 \end{bmatrix}} + \overset{c_2}{\begin{bmatrix} 3 \\ -1 \end{bmatrix}} \overset{r_2}{\begin{bmatrix} -3 & 5 & 1 \end{bmatrix}} \\
&= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix}
\end{aligned}
\tag{13}
$$

The main use of the column-row expansion is for developing theoretical results rather than for numerical computations.

As a check, we leave it for you to confirm that the product in (12) and the sum in (13) both yield

$$AB = \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$ ◀

Matrix Form of a Linear System

Matrix multiplication has an important application to systems of linear equations. Consider a system of m linear equations in n unknowns:

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m
\end{array}
$$

$5x+4y=3$

$x-2=0$

$$\begin{bmatrix} 5 & 4 & \vdots & 3 \\ 1 & -2 & \vdots & 0 \end{bmatrix}$$

augmented matrix

2×3

coefficient matrix

$$\begin{bmatrix} 5 & 4 \\ 1 & -2 \end{bmatrix}$$

2×2

Since two matrices are equal if and only if their corresponding entries are equal, we can replace the m equations in this system by the single matrix equation

$$
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

The $m \times 1$ matrix on the left side of this equation can be written as a product to give

$$
\underset{m\times n}{\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}}
\underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}
=
\underset{m\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}
$$

If we designate these matrices by A, $\mathbf{x}$, and $\mathbf{b}$, respectively, then we can replace the original system of m equations in n unknowns by the single matrix equation

$$A\mathbf{x} = \mathbf{b}$$

The matrix A in this equation is called the ***coefficient matrix*** of the system. The augmented matrix for the system is obtained by adjoining $\mathbf{b}$ to A as the last column; thus the augmented matrix is

$$
[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]
$$

The vertical partition line in the augmented matrix $[A \mid \mathbf{b}]$ is optional, but is a useful way of visually separating the coefficient matrix A from the column vector $\mathbf{b}$.

Transpose of a Matrix

We conclude this section by defining two matrix operations that have no analogs in the arithmetic of real numbers.

coefficient matr

DEFINITION 7 If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

EXAMPLE 11 Some Transposes

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

3x4 3x2 1x3 1x1

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

4x3 2x3 3x1 1x1

$D = D^T$

Observe that not only are the columns of A^T the rows of A, but the rows of A^T are the columns of A. Thus the entry in row i and column j of A^T is the entry in row j and column i of A; that is,

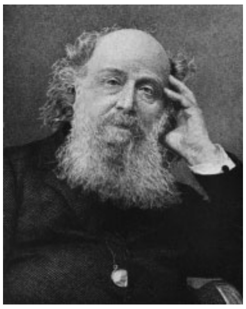
$$(A^T)_{ij} = (A)_{ji} \tag{14}$$

Note the reversal of the subscripts.

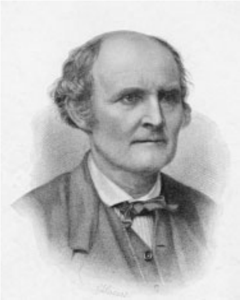
In the special case where A is a square matrix, the transpose of A can be obtained by interchanging entries that are symmetrically positioned about the main diagonal. In (15) we see that A^T can also be obtained by "reflecting" A about its main diagonal.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix} \tag{15}$$

Interchange entries that are symmetrically positioned about the main diagonal.



James Sylvester (1814–1897)



Arthur Cayley (1821–1895)

Historical Note The term *matrix* was first used by the English mathematician James Sylvester, who defined the term in 1850 to be an "oblong arrangement of terms." Sylvester communicated his work on matrices to a fellow English mathematician and lawyer named Arthur Cayley, who then introduced some of the basic operations on matrices in a book entitled *Memoir on the Theory of Matrices* that was published in 1858. As a matter of interest, Sylvester, who was Jewish, did not get his college degree because he refused to sign a required oath to the Church of England. He was appointed to a chair at the University of Virginia in the United States but resigned after swatting a student with a stick because he was reading a newspaper in class. Sylvester, thinking he had killed the student, fled back to England on the first available ship. Fortunately, the student was not dead, just in shock!

[*Images:* © *Bettmann/CORBIS* (*Sylvester*); *Photo Researchers/Getty Images* (*Cayley*)]

Trace of a Matrix

DEFINITION 8 If A is a square matrix, then the ***trace of A***, denoted by $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

$$C = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\mathrm{tr}(C) = 1 + (-1) + 2 = \boxed{2}$$

EXAMPLE 12 Trace

The following are examples of matrices and their traces.

$$\overset{3\times 3}{A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}, \quad \overset{4\times 4}{B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}}$$

$$\mathrm{tr}(A) = \underline{a_{11}} + \underline{a_{22}} + \underline{a_{33}} \qquad \mathrm{tr}(B) = -1 + 5 + 7 + 0 = 11$$

In the exercises you will have some practice working with the transpose and trace operations.

Exercise Set 1.3

In Exercises 1–2, suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

1. (a) BA (b) AB^T (c) $AC + D$

(d) $E(AC)$ (e) $A - 3E^T$ (f) $E(5B + A)$

2. (a) CD^T (b) DC (c) $BC - 3D$

(d) $D^T(BE)$ (e) $B^TD + ED$ (f) $BA^T + D$

In Exercises 3–6, use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

3. (a) $D + E$ (b) $D - E$ (c) $5A$

(d) $-7C$ (e) $2B - C$ (f) $4E - 2D$

(g) $-3(D + 2E)$ (h) $A - A$ (i) $\mathrm{tr}(D)$

(j) $\mathrm{tr}(D - 3E)$ (k) $4\,\mathrm{tr}(7B)$ (l) $\mathrm{tr}(A)$

4. (a) $2A^T + C$ (b) $D^T - E^T$ (c) $(D - E)^T$

(d) $B^T + 5C^T$ (e) $\frac{1}{2}C^T - \frac{1}{4}A$ (f) $B - B^T$

(g) $2E^T - 3D^T$ (h) $(2E^T - 3D^T)^T$ (i) $(CD)E$

(j) $C(BA)$ (k) $\mathrm{tr}(DE^T)$ (l) $\mathrm{tr}(BC)$

5. (a) AB (b) BA (c) $(3E)D$

(d) $(AB)C$ (e) $A(BC)$ (f) CC^T

(g) $(DA)^T$ (h) $(C^TB)A^T$ (i) $\mathrm{tr}(DD^T)$

(j) $\mathrm{tr}(4E^T - D)$ (k) $\mathrm{tr}(C^TA^T + 2E^T)$ (l) $\mathrm{tr}((EC^T)^TA)$

6. (a) $(2D^T - E)A$ (b) $(4B)C + 2B$

(c) $(-AC)^T + 5D^T$ (d) $(BA^T - 2C)^T$

(e) $B^T(CC^T - A^TA)$ (f) $D^TE^T - (ED)^T$

In Exercises 7–8, use the following matrices and either the row method or the column method, as appropriate, to find the indicated row or column.

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

7. (a) the first row of AB (b) the third row of AB

(c) the second column of AB (d) the first column of BA

(e) the third row of AA (f) the third column of AA

8. (a) the first column of AB (b) the third column of BB

(c) the second row of BB (d) the first column of AA

(e) the third column of AB (f) the first row of BA

▶ In Exercises 9–10, use matrices A and B from Exercises **7–8**. ◀

9. (a) Express each column vector of AA as a linear combination of the column vectors of A.

(b) Express each column vector of BB as a linear combination of the column vectors of B.

10. (a) Express each column vector of AB as a linear combination of the column vectors of A.

(b) Express each column vector of BA as a linear combination of the column vectors of B.

▶ In each part of Exercises 11–12, find matrices A, $\mathbf{x}$, and $\mathbf{b}$ that express the given linear system as a single matrix equation $A\mathbf{x} = \mathbf{b}$, and write out this matrix equation. ◀

11. (a)
$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 - x_2 + x_3 &= -1 \\ x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 4x_1 \qquad\quad - 3x_3 + x_4 &= 1 \\ 5x_1 + x_2 \qquad\quad - 8x_4 &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 &= 0 \\ 3x_2 - x_3 + 7x_4 &= 2 \end{aligned}$$

12. (a)
$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -3 \\ 2x_1 + x_2 \qquad\quad &= 0 \\ - 3x_2 + 4x_3 &= 1 \\ x_1 \qquad\quad + x_3 &= 5 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 + 3x_2 + 3x_3 &= -3 \\ -x_1 - 5x_2 - 2x_3 &= 3 \\ - 4x_2 + x_3 &= 0 \end{aligned}$$

▶ In each part of Exercises 13–14, express the matrix equation as a system of linear equations. ◀

13. (a)
$$\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$$

14. (a)
$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

▶ In Exercises 15–16, find all values of k, if any, that satisfy the equation. ◀

15.
$$\begin{bmatrix} k & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$$

16.
$$\begin{bmatrix} 2 & 2 & k \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$$

▶ In Exercises 17–20, use the column-row expansion of AB to express this product as a sum of matrices. ◀

17. $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 0 & -2 \\ 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 1 \\ -3 & 0 & 2 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

20. $A = \begin{bmatrix} 0 & 4 & 2 \\ 1 & -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ 1 & -1 \end{bmatrix}$

21. For the linear system in Example 5 of Section 1.2, express the general solution that we obtained in that example as a linear combination of column vectors that contain only numerical entries. [*Suggestion:* Rewrite the general solution as a single column vector, then write that column vector as a sum of column vectors each of which contains at most one parameter, and then factor out the parameters.]

22. Follow the directions of Exercise 21 for the linear system in Example 6 of Section 1.2.

▶ In Exercises 23–24, solve the matrix equation for a, b, c, and d. ◀

23.
$$\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$$

24.
$$\begin{bmatrix} a-b & b+a \\ 3d+c & 2d-c \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

25. (a) Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.

(b) Find a similar result involving a column of zeros.

26. In each part, find a 6×6 matrix $[a_{ij}]$ that satisfies the stated condition. Make your answers as general as possible by using letters rather than specific numbers for the nonzero entries.

(a) $a_{ij} = 0$ if $i \neq j$ (b) $a_{ij} = 0$ if $i > j$

(c) $a_{ij} = 0$ if $i < j$ (d) $a_{ij} = 0$ if $|i - j| > 1$

▶ In Exercises 27–28, how many 3×3 matrices A can you find for which the equation is satisfied for all choices of x, y, and z? ◀

27. $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$

28. $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix}$

29. A matrix B is said to be a ***square root*** of a matrix A if $BB = A$.

(a) Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) How many different square roots can you find of $A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$?

(c) Do you think that every 2×2 matrix has at least one square root? Explain your reasoning.

30. Let 0 denote a 2×2 matrix, each of whose entries is zero.

(a) Is there a 2×2 matrix A such that $A \neq 0$ and $AA = 0$? Justify your answer.

(b) Is there a 2×2 matrix A such that $A \neq 0$ and $AA = A$? Justify your answer.

31. Establish Formula (11) by using Formula (5) to show that

$$(AB)_{ij} = (\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r)_{ij}$$

32. Find a 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

(a) $a_{ij} = i + j$

(b) $a_{ij} = i^{j-1}$

(c) $a_{ij} = \begin{cases} 1 & \text{if} \quad |i-j| > 1 \\ -1 & \text{if} \quad |i-j| \leq 1 \end{cases}$

33. Suppose that type I items cost \$1 each, type II items cost \$2 each, and type III items cost \$3 each. Also, suppose that the accompanying table describes the number of items of each type purchased during the first four months of the year.

Table Ex-33

	Type I	Type II	Type III
Jan.	3	4	3
Feb.	5	6	0
Mar.	2	9	4
Apr.	1	1	7

What information is represented by the following product?

$$\begin{bmatrix} 3 & 4 & 3 \\ 5 & 6 & 0 \\ 2 & 9 & 4 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

34. The accompanying table shows a record of May and June unit sales for a clothing store. Let M denote the 4×3 matrix of May sales and J the 4×3 matrix of June sales.

(a) What does the matrix $M + J$ represent?

(b) What does the matrix $M - J$ represent?

(c) Find a column vector $\mathbf{x}$ for which $M\mathbf{x}$ provides a list of the number of shirts, jeans, suits, and raincoats sold in May.

(d) Find a row vector $\mathbf{y}$ for which $\mathbf{y}M$ provides a list of the number of small, medium, and large items sold in May.

(e) Using the matrices $\mathbf{x}$ and $\mathbf{y}$ that you found in parts (c) and (d), what does $\mathbf{y}M\mathbf{x}$ represent?

Table Ex-34

May Sales

	Small	Medium	Large
Shirts	45	60	75
Jeans	30	30	40
Suits	12	65	45
Raincoats	15	40	35

June Sales

	Small	Medium	Large
Shirts	30	33	40
Jeans	21	23	25
Suits	9	12	11
Raincoats	8	10	9

Working with Proofs

35. Prove: If A and B are $n \times n$ matrices, then

$$\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$$

36. (a) Prove: If AB and BA are both defined, then AB and BA are square matrices.

(b) Prove: If A is an $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

True-False Exercises

TF. In parts (a)–(o) determine whether the statement is true or false, and justify your answer.

(a) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has no main diagonal.

(b) An $m \times n$ matrix has m column vectors and n row vectors.

(c) If A and B are 2×2 matrices, then $AB = BA$.

(d) The ith row vector of a matrix product AB can be computed by multiplying A by the ith row vector of B.