جامعة الملك سعود
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STAT415
probability(2)

part 7 –problems from SOA

191. A fire in an apartment building results in a loss, X, to the owner and a loss, Y, to the tenants. The variables X and Y have a bivariate normal distribution with $E(X) = 40$, $\text{Var}(X) = 76$, $E(Y) = 30$, $\text{Var}(Y) = 32$, and $\text{Var}(X \mid Y = 28.5) = 57$.

Calculate $\text{Var}(Y \mid X = 25)$.

(A) 13
(B) 24
(C) 32
(D) 50
(E) 57

Let the random variables X and Y have the bivariate normal distribution. The conditional distributions of X and Y are

$$Y \mid X = x \sim N\left(\mu_Y + \frac{\rho\sigma_Y(x - \mu_X)}{\sigma_X}, (1 - \rho^2)\,\sigma_Y^2\right)$$

and

$$X \mid Y = y \sim N\left(\mu_X + \frac{\rho\sigma_X(y - \mu_Y)}{\sigma_Y}, (1 - \rho^2)\,\sigma_X^2\right)$$

191. Solution: B

$$57 = Var(X \mid Y = 28.5) = (1-\rho^2)Var(X) = (1-\rho^2)76$$

$$1-\rho^2 = 57/76 = 0.75$$

$$Var(Y \mid X = 25) = (1-\rho^2)Var(Y) = 0.75(32) = 24$$

242. The annual profits that company A and company B earn follow a bivariate normal distribution.

Company A's annual profit has mean 2000 and standard deviation 1000.

Company B's annual profit has mean 3000 and standard deviation 500.

The correlation coefficient between these annual profits is 0.80.

Calculate the probability that company B's annual profit is less than 3900, given that company A's annual profit is 2300.

(A) 0.8531
(B) 0.9192
(C) 0.9641
(D) 0.9744
(E) 0.9953

242. Solution: E

Let X and Y represent the annual profits for companies A and B, respectively.

We are given that X and Y have a bivariate normal distribution, the correlation coefficient is $\rho = 0.8$, X has mean $\mu_X = 2000$ and standard deviation $\sigma_X = 1000$, and Y has mean $\mu_Y = 3000$ and standard deviation $\sigma_Y = 500$.

In general for a bivariate normal distribution, given that $X = x$, Y is normally distributed with mean $\mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - \mu_X)$ and standard deviation $\sigma_Y\sqrt{1-\rho^2}$.

So given that company A's annual profit is 2300, company B's annual profit is normally distributed with mean $3000 + \frac{0.8(500)}{1000}(2300 - 2000) = 3120$ and standard deviation $500\sqrt{1-(0.8)^2} = 300$.

Therefore, given that company A's annual profit is 2300, the probability that company B's profit is at most 3900 is $\mathrm{P}\left[Z \le \frac{3900 - 3120}{300}\right] = \mathrm{P}[Z \le 2.6] = 0.9953$.

71. The time, T, that a manufacturing system is out of operation has cumulative distribution function

$$F(t) = \begin{cases} 1-\left(\frac{2}{t}\right)^2, & t > 2 \\ 0, & \text{otherwise.} \end{cases}$$

The resulting cost to the company is $Y = T^2$. Let g be the density function for Y.

Determine $g(y)$, for $y > 4$.

(A) $\frac{4}{y^2}$

(B) $\frac{8}{y^{3/2}}$

(C) $\frac{8}{y^3}$

(D) $\frac{16}{y}$

(E) $\frac{1024}{y^5}$

71. Solution: A

The distribution function of Y is given by $G(y) = P(T^2 \leq y) = P(T \leq \sqrt{y}) = F(\sqrt{y}) = 1 - 4/y$ for $y > 4$. Differentiate to obtain the density function $g(y) = 4y^{-2}$.
Alternatively, the density function of T $f(t) = F'(t) = 8t^{-3}$. We have $t = y^{0.5}$ and $dt = 0.5y^{-0.5}dy$.
Then $g(y) = f(y^{0.5})\left|dt/dy\right| = 8(y^{0.5})^{-3}(0.5y^{-0.5}) = 4y^{-2}$.

72. An investment account earns an annual interest rate R that follows a uniform distribution on the interval (0.04, 0.08). The value of a 10,000 initial investment in this account after one year is given by $V = 10,000e^{R}$.

Let F be the cumulative distribution function of V.

Determine $F(v)$ for values of v that satisfy $0 < F(v) < 1$.

(A) $\dfrac{10,000e^{v/10,000} - 10,408}{425}$

(B) $25e^{v/10,000} - 0.04$

(C) $\dfrac{v - 10,408}{10,833 - 10,408}$

(D) $\dfrac{25}{v}$

(E) $25\left[\ln\left(\dfrac{v}{10,000}\right) - 0.04\right]$

72. Solution: E

The distribution function of V is given by

$$F(v) = P[V \le v] = P\left[10{,}000e^R \le v\right] = P\left[R \le \ln(v) - \ln(10{,}000)\right]$$

$$= \int_{0.04}^{\ln(v)-\ln(10{,}000)} \frac{1}{0.04}\,dr = \frac{r}{0.04}\Bigg|_{0.04}^{\ln(v)-\ln(10{,}000)} = 25\ln(v) - 25\ln(10{,}000) - 1$$

$$= 25\left[\ln\left(\frac{v}{10{,}000}\right) - 0.04\right].$$

73. An actuary models the lifetime of a device using the random variable $Y = 10X^{0.8}$, where X is an exponential random variable with mean 1.

Let $f(y)$ be the density function for Y.

Determine $f(y)$, for $y > 0$.

(A) $10y^{0.8}\exp(-8y^{-0.2})$

(B) $8y^{-0.2}\exp(-10y^{0.8})$

(C) $8y^{-0.2}\exp[-(0.1y)^{1.25}]$

(D) $(0.1y)^{1.25}\exp[-0.125(0.1y)^{0.25}]$

(E) $0.125(0.1y)^{0.25}\exp[-(0.1y)^{1.25}]$

73. Solution: E

$$F(y)=P[Y\le y]=P[10X^{0.8}\le y]=\Pr\left[X\le(0.1y)^{1.25}\right]=1-e^{-(0.1y)^{1.25}}.$$

Therefore, $f(y)=F'(y)=0.125(0.1y)^{0.25}e^{-(0.1y)^{1.25}}$.

74. Let T denote the time in minutes for a customer service representative to respond to 10 telephone inquiries. T is uniformly distributed on the interval with endpoints 8 minutes and 12 minutes.

Let R denote the average rate, in customers per minute, at which the representative responds to inquiries, and let $f(r)$ be the density function for R.

Determine $f(r)$, for $\frac{10}{12} \le r \le \frac{10}{8}$.

(A) $\frac{12}{5}$

(B) $3 - \frac{5}{2r}$

(C) $3r - \frac{5\ln(r)}{2}$

(D) $\frac{10}{r^2}$

(E) $\frac{5}{2r^2}$

74. Solution: E

First note $R = 10/T$. Then,

$F(r) = P[R \le r] = P[10/T \le r] = P[T \ge 10/r] = \int_{10/r}^{12} 0.25dt = 0.25(12 - 10/r)$. The density function is $f(r) = F'(r) = 2.5/r^2$.

Alternatively, $t = 10/r$ and $dt = -10/r^2 dr$. Then $f_R(r) = f_T(10/r)\,|dt/dr| = 0.25(10/\mathrm{r}^2) = 2.5/r^2$.

75. The monthly profit of Company I can be modeled by a continuous random variable with density function f. Company II has a monthly profit that is twice that of Company I.

Let g be the density function for the distribution of the monthly profit of Company II.

Determine $g(x)$ where it is not zero.

(A) $\frac{1}{2} f\left(\frac{x}{2}\right)$

(B) $f\left(\frac{x}{2}\right)$

(C) $2 f\left(\frac{x}{2}\right)$

(D) $2 f(x)$

(E) $2 f(2x)$

75. Solution: A

Let Y be the profit for Company II, so $Y = 2X$.

$$F_Y(y) = P[Y \le y] = P[2X \le y] = P[X \le y/2] = F_X(y/2)$$

$$f_Y(y) = F_Y'(y) = F_X'(y/2) = (1/2) f_X(y/2).$$

Alternatively, $X = Y/2$ and $dx = dy/2$. Then,

$$f_Y(y) = f_X(y/2)\,|dy/dx| = f_X(y/2)(1/2).$$

183. X is a random variable with probability density function

$$f(x) = \begin{cases} e^{-2x}, & x \geq 0 \\ 2e^{4x}, & x < 0 \end{cases}$$

Let $T = X^2$.

Determine the probability density function for T for positive values of t.

(A) $f(t) = \dfrac{e^{-2\sqrt{t}}}{2\sqrt{t}} + \dfrac{e^{-4\sqrt{t}}}{\sqrt{t}}$

(B) $f(t) = \dfrac{e^{-2\sqrt{t}}}{2\sqrt{t}}$

(C) $f(t) = e^{-2t} + 2e^{-4t}$

(D) $f(t) = 2te^{-2t^2} + 4te^{-4t^2}$

(E) $f(t) = 2te^{-2t^2}$

183. Solution: A

$$F(t) = P(T \le t) = P(X^2 \le t) = P\left(-\sqrt{t} \le X \le \sqrt{t}\right)$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} f(x)dx = \int_{-\sqrt{t}}^{0} 2e^{4x}dx + \int_{0}^{\sqrt{t}} e^{-2x} = 0.5e^{4x}\Big|_{-\sqrt{t}}^{0} - 0.5e^{-2x}\Big|_{0}^{\sqrt{t}} = 0.5 - 0.5e^{-4\sqrt{t}} - 0.5e^{-2\sqrt{t}} + 0.5$$

$$= 1 - 0.5e^{-4\sqrt{t}} - 0.5e^{-2\sqrt{t}}$$

$$f(t) = F'(t) = -0.5e^{-4\sqrt{t}}[-4(0.5)/\sqrt{t}] - 0.5e^{-2\sqrt{t}}[-2(0.5)/\sqrt{t}] = e^{-4\sqrt{t}}/\sqrt{t} + 0.5e^{-2\sqrt{t}}/\sqrt{t}$$

$$= \frac{e^{-2\sqrt{t}}}{2\sqrt{t}} + \frac{e^{-4\sqrt{t}}}{\sqrt{t}}$$