STAT415
probability(2)

part 6 –problems from SOA

77. A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

$$f(x,y) = \frac{x+y}{8}, \text{ for } 0 < x < 2 \text{ and } 0 < y < 2.$$

Calculate the probability that the device fails during its first hour of operation.

(A) 0.125
(B) 0.141
(C) 0.391
(D) 0.625
(E) 0.875

77. Solution: D

The probability it works for at least one hour is the probability that both components work for more than one hour. This probability is

$$\int_1^2 \int_1^2 \frac{x+y}{8} dxdy = \int_1^2 \frac{0.5x^2 + xy}{8}\bigg|_1^2 dy = \int_1^2 \frac{1.5+y}{8} dy = \frac{1.5y + 0.5y^2}{8}\bigg|_1^2 = 0.375.$$

The probability of failing within one hour is the complement, 0.625.

79. A device contains two components. The device fails if either component fails. The joint density function of the lifetimes of the components, measured in hours, is $f(s,t)$, where $0 < s < 1$ and $0 < t < 1$.

Determine which of the following represents the probability that the device fails during the first half hour of operation.

(A) $\int_0^{0.5}\int_0^{0.5} f(s,t)\,ds\,dt$

(B) $\int_0^{1}\int_0^{0.5} f(s,t)\,ds\,dt$

(C) $\int_{0.5}^{1}\int_{0.5}^{1} f(s,t)\,ds\,dt$

(D) $\int_0^{0.5}\int_0^{1} f(s,t)\,ds\,dt + \int_0^{1}\int_0^{0.5} f(s,t)\,ds\,dt$

(E) $\int_0^{0.5}\int_{0.5}^{1} f(s,t)\,ds\,dt + \int_0^{1}\int_0^{0.5} f(s,t)\,ds\,dt$

79. Solution: E

Let s be on the horizontal axis and *t* be on the vertical axis. The event in question covers all but the upper right quarter of the unit square. The probability is the integral over the other three quarters. Answer (A) is the lower left quarter. Answer B is the left half. Answer (C) is the upper right quarter. Answer (D) is the lower half plus the left half, so the lower left quarter is counted twice. Answer (E) is the lower right corner plus the left half, which is the correct region.

For this question, the regions don't actually need to be identified. The area is 0.75 while the five answer choices integrate over regions of area 0.25, 0.5, 0.25, 1, and 0.75 respectively. So only (E) can be correct.

89. The future lifetimes (in months) of two components of a machine have the following joint density function:

$$f(x,y)=\begin{cases}\dfrac{6}{125{,}000}(50-x-y), & 0<x<50-y<50\\ 0, & \text{otherwise.}\end{cases}$$

Determine which of the following represents the probability that both components are still functioning 20 months from now.

(A) $\dfrac{6}{125{,}000}\int_{0}^{20}\int_{0}^{20}(50-x-y)\,dy\,dx$

(B) $\dfrac{6}{125{,}000}\int_{20}^{30}\int_{20}^{50-x}(50-x-y)\,dy\,dx$

(C) $\dfrac{6}{125{,}000}\int_{20}^{30}\int_{20}^{50-x-y}(50-x-y)\,dy\,dx$

(D) $\dfrac{6}{125{,}000}\int_{20}^{50}\int_{20}^{50-x}(50-x-y)\,dy\,dx$

(E) $\dfrac{6}{125{,}000}\int_{20}^{50}\int_{20}^{50-x-y}(50-x-y)\,dy\,dx$

89. Solution: B

The probability both variables exceed 20 is represented by the triangle with vertices (20,20), (20,30), and (30,20). All the answer choices have x as the outer integral and x ranges from 20 to 30, eliminating answers (A), (D), and (E). For a given value of x, the triangle runs from the base at $y = 20$ to the diagonal line at $y = 50 - x$. This is answer (B).

91. An insurance company insures a large number of drivers. Let X be the random variable representing the company's losses under collision insurance, and let Y represent the company's losses under liability insurance. X and Y have joint density function

$$f(x,y)=\begin{cases}\dfrac{2x+2-y}{4}, & 0<x<1 \text{ and } 0<y<2\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the probability that the total company loss is at least 1.

(A) 0.33
(B) 0.38
(C) 0.41
(D) 0.71
(E) 0.75

91. Solution: D

$$P[X+Y \geq 1] = \int_0^1 \int_{1-x}^{2} \frac{2x+2-y}{4} \, dy dx = \int_0^1 \left. \frac{2xy+2y-0.5y^2}{4} \right|_{1-x}^{2} dx$$

$$= \int_0^1 \frac{4x+4-2}{4} - \frac{2x(1-x)+2(1-x)-0.5(1-x)^2}{4} \, dx$$

$$= \int_0^1 \frac{2.5x^2+3x+0.5}{4} \, dx = \frac{2.5/3+3/2+0.5}{4} = 0.708.$$

104. A joint density function is given by

$$f(x,y)=\begin{cases} kx, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise.} \end{cases}$$

where k is a constant.

Calculate Cov(X,Y).

(A) –1/6
(B) 0
(C) 1/9
(D) 1/6
(E) 2/3

104. Solution: B

First determine k:

$$1=\int_0^1\int_0^1 kxdxdy=\int_0^1 0.5kx^2\Big|_0^1 dy=\int_0^1 0.5kdy=0.5k$$

$k=2.$

Then

$$E[X]=\int_0^1\int_0^1 2x^2dydx=\int_0^1 2x^2dx=\frac{2}{3}$$

$$E[Y]=\int_0^1\int_0^1 y2x\ dxdy=\int_0^1 ydy=\frac{1}{2}$$

$$E[XY]=\int_0^1\int_0^1 2x^2ydxdy=\int_0^1 (2/3)ydy=\frac{1}{3}$$

$$\text{Cov}(X,Y)=E[XY]-E[X]E[Y]=\frac{1}{3}-\frac{2}{3}\frac{1}{2}=0.$$

Alternatively, note that the density function factors as the product of one term that depends only on x and one that depends only on y. Therefore, the two variables are independent and the covariance must be 0.

105. Let X and Y be continuous random variables with joint density function

$$f(x,y)=\begin{cases}\frac{8}{3}xy, & 0\le x\le 1,\ x\le y\le 2x\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the covariance of X and Y.

(A) 0.04
(B) 0.25
(C) 0.67
(D) 0.80
(E) 1.24

105. Solution: A

Note that although the density function factors into expressions involving only *x* and *y*, the variables are not independent. An additional requirement is that the domain be a rectangle.

$$E[X]=\int_0^1\int_x^{2x}\frac{8}{3}x^2y\,dy\,dx=\int_0^1\frac{4}{3}x^2y^2\Big|_x^{2x}dx=\int_0^1\frac{4}{3}x^2\left(4x^2-x^2\right)dx=\int_0^1 4x^4dx=\frac{4}{5}x^5\Big|_0^1=\frac{4}{5}$$

$$E[Y]=\int_0^1\int_x^{2x}\frac{8}{3}xy^2dy\,dx=\int_0^1\frac{8}{9}xy^3\Big|_x^{2x}dx=\int_0^1\frac{8}{9}x\left(8x^3-x^3\right)dx=\int_0^1\frac{56}{9}x^4dx=\frac{56}{45}x^5\Big|_0^1=\frac{56}{45}$$

$$E[XY]=\int_0^1\int_x^{2x}\frac{8}{3}x^2y^2dy\,dx=\int_0^1\frac{8}{9}x^2y^3\Big|_x^{2x}dx=\int_0^1\frac{8}{9}x^2\left(8x^3-x^3\right)dx=\int_0^1\frac{56}{9}x^5dx=\frac{56}{54}=\frac{28}{27}$$

$$\text{Cov}(X,Y)=E[XY]-E[X]E[Y]=\frac{28}{27}-\left(\frac{56}{45}\right)\left(\frac{4}{5}\right)=0.04.$$

106. Let X and Y denote the values of two stocks at the end of a five-year period. X is uniformly distributed on the interval (0, 12). Given $X = x$, Y is uniformly distributed on the interval $(0, x)$.

Calculate Cov(X, Y) according to this model.

(A) 0
(B) 4
(C) 6
(D) 12
(E) 24

106. Solution: C

The joint density function of X and Y is

$$f(x,y) = f(x)f(y\,|\,x) = \frac{1}{12}\frac{1}{x} = \frac{1}{12x},\; 0 < x < 12,\; 0 < y < x.$$

$$E[X] = \int_0^{12}\int_0^{x} \frac{x}{12x}\,dydx = \int_0^{12} \frac{y}{12}\Big|_0^x dx = \int_0^{12} \frac{x}{12}\,dx = \frac{x^2}{24}\Big|_0^{12} = \frac{144}{24}$$

$$E[Y] = \int_0^{12}\int_0^{x} \frac{y}{12x}\,dydx = \int_0^{12} \frac{y^2}{24x}\Big|_0^x dx = \int_0^{12} \frac{x}{24}\,dx = \frac{x^2}{48}\Big|_0^{12} = \frac{144}{48}$$

$$E[XY] = \int_0^{12}\int_0^{x} \frac{xy}{12x}\,dydx = \int_0^{12} \frac{y^2}{24}\Big|_0^x dx = \int_0^{12} \frac{x^2}{24}\,dx = \frac{x^3}{72}\Big|_0^{12} = \frac{1728}{72}$$

$$Cov(X,Y) = \frac{1728}{72} - \frac{144}{24}\frac{144}{48} = 6.$$

110. Let X and Y be continuous random variables with joint density function

$$f(x,y)=\begin{cases}24xy, & 0<x<1, 0<y<1-x\\ 0, & \text{otherwise.}\end{cases}$$

Calculate $\mathrm{P}\left[Y<X \middle| X=\frac{1}{3}\right]$.

(A) 1/27
(B) 2/27
(C) 1/4
(D) 1/3
(E) 4/9

110. Solution: C

The calculations are:

$$f(y \mid x = 1/3) = \frac{f(1/3, y)}{f_X(1/3)}, \quad 0 < y < \frac{2}{3}$$

$$f_X(1/3) = \int_0^{2/3} 24(1/3)y\,dy = 8(2/3)^2/2 = 16/9$$

$$f(y \mid x = 1/3) = \frac{8y}{16/9} = 4.5y, \quad 0 < y < \frac{2}{3}$$

$$P[Y < X \mid X = 1/3] = P[Y < 1/3 \mid X = 1/3] = \int_0^{1/3} 4.5y\,dy = 4.5(1/3)^2/2 = 1/4.$$

111. Once a fire is reported to a fire insurance company, the company makes an initial estimate, X, of the amount it will pay to the claimant for the fire loss. When the claim is finally settled, the company pays an amount, Y, to the claimant. The company has determined that X and Y have the joint density function

$$f(x,y)=\begin{cases}\dfrac{2}{x^2(x-1)}y^{-(2x-1)/(x-1)}, & x>1,\ y>1\\ 0, & \text{otherwise.}\end{cases}$$

Given that the initial claim estimated by the company is 2, calculate the probability that the final settlement amount is between 1 and 3.

(A) 1/9
(B) 2/9
(C) 1/3
(D) 2/3
(E) 8/9

111. Solution: E

$$P[1 < Y < 3 \mid X = 2] = \int_1^3 \frac{f(2,y)}{f_X(2)} dy$$

$$f(2,y) = \frac{2}{2^2(2-1)} y^{-(4-1)/(2-1)} = 0.5y^{-3}$$

$$f_X(2) = \int_1^\infty f(2,y)dy = \int_1^\infty 0.5y^{-3}dy = -0.25y^{-2}\Big|_1^\infty = 0.25$$

$$P[1 < Y < 3 \mid X = 2] = \int_1^3 \frac{0.5y^{-3}}{0.25} dy = -y^{-2}\Big|_1^3 = -1/9 + 1 = 8/9.$$

112. A company offers a basic life insurance policy to its employees, as well as a supplemental life insurance policy. To purchase the supplemental policy, an employee must first purchase the basic policy.

Let X denote the proportion of employees who purchase the basic policy, and Y the proportion of employees who purchase the supplemental policy. Let X and Y have the joint density function $f(x,y) = 2(x + y)$ on the region where the density is positive.

Given that 10% of the employees buy the basic policy, calculate the probability that fewer than 5% buy the supplemental policy.

(A) 0.010
(B) 0.013
(C) 0.108
(D) 0.417
(E) 0.500

112. Solution: D

Because only those with the basic policy can purchase the supplemental policy, $0 < y < x < 1$. Then,

$$P[Y < 0.05 \mid X = 0.10] = \int_0^{0.05} \frac{f(0.10, y)}{f_X(0.10)} dy$$

$$f(0.10, y) = 2(0.10 + y), 0 < y < 0.10$$

$$f_X(0.10) = \int_0^{0.10} f(0.10, y)dy = \int_0^{0.10} 2(0.10 + y)dy = 0.2y + y^2 \Big|_0^{0.10} = 0.03$$

$$P[Y < 0.05 \mid X = 0.10] = \int_0^{0.05} \frac{2(0.10 + y)}{0.03} dy = \frac{0.2y + y^2}{0.03} \Bigg|_0^{0.05} = \frac{0.01 + 0.0025}{0.03} = 0.417.$$

115. The stock prices of two companies at the end of any given year are modeled with random variables X and Y that follow a distribution with joint density function

$$f(x,y)=\begin{cases} 2x, & 0<x<1,\ x<y<x+1 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the conditional variance of Y given that $X = x$.

(A) 1/12
(B) 7/6
(C) $x + 1/2$
(D) $x^2 - 1/6$
(E) $x^2 + x + 1/3$

115. Solution: A

$$f(y \mid x) = \frac{f(x, y)}{f_X(x)}$$

$$f_X(x) = \int_x^{x+1} 2x dy = 2x$$

$$f(y \mid x) = \frac{2x}{2x} = 1, x < y < x + 1.$$

The conditional variance is uniform on the interval $(x, x + 1)$. A uniform random variable on a unit interval has variance 1/12. Alternatively the integrals can be done to obtain the mean of x + 0.5 and second moment of $x^2 + x + 1/3$. The second moment minus the square of them mean gives the variance of 1/12.

117. A company is reviewing tornado damage claims under a farm insurance policy. Let X be the portion of a claim representing damage to the house and let Y be the portion of the same claim representing damage to the rest of the property. The joint density function of X and Y is

$$f(x,y)=\begin{cases}6[1-(x+y)], & x>0,\ y>0,\ x+y<1\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the probability that the portion of a claim representing damage to the house is less than 0.2.

(A) 0.360
(B) 0.480
(C) 0.488
(D) 0.512
(E) 0.520

117. Solution: C

The marginal density of X is

$$f_X(x) = \int_0^{1-x} 6(1-x-y)dy = 6(y - xy - 0.5y^2)\Big|_0^{1-x} = 6[1-x-x(1-x)-0.5(1-x)^2]$$

$$= 6[1-x-x+x^2-0.5+x-0.5x^2] = 6(0.5x^2-x+0.5),\ 0<x<1.$$

The requested probability is

$$P[X<0.2] = \int_0^{0.2} 6(0.5x^2-x+0.5)dx = x^3-3x^2+3x\Big|_0^{0.2} = 0.008-0.12+0.6 = 0.488.$$

118. Let X and Y be continuous random variables with joint density function

$$f(x,y)=\begin{cases}15y, & x^2\le y\le x\\ 0, & \text{otherwise.}\end{cases}$$

Let g be the marginal density function of Y.

Determine which of the following represents g.

(A) $g(y)=\begin{cases}15y, & 0<y<1\\ 0, & \text{otherwise}\end{cases}$

(B) $g(y)=\begin{cases}\dfrac{15y^2}{2}, & x^2<y<x\\ 0, & \text{otherwise}\end{cases}$

(C) $g(y)=\begin{cases}\dfrac{15y^2}{2}, & 0<y<1\\ 0, & \text{otherwise}\end{cases}$

(D) $g(y)=\begin{cases}15y^{3/2}(1-y^{1/2}), & x^2<y<x\\ 0, & \text{otherwise}\end{cases}$

(E) $g(y)=\begin{cases}15y^{3/2}(1-y^{1/2}), & 0<y<1\\ 0, & \text{otherwise}\end{cases}$

118. Solution: E

$$g(y)=\int_y^{\sqrt{y}} 15y\,dx = 15yx\Big|_y^{\sqrt{y}} = 15y(\sqrt{y}-y) = 15y^{3/2}(1-y^{1/2}),\ 0<y<1$$

The limits are found by noting that x must be less than the square root of y and also must be greater than y. While not directly stated, the only values of x for which the square is smaller are $0 < x < 1$. This implies y is constrained to the same range and thus its square root must be larger, ensuring that the integral has the correct sign.

121. Let X represent the age of an insured automobile involved in an accident. Let Y represent the length of time the owner has insured the automobile at the time of the accident.

X and Y have joint probability density function

$$f(x,y)=\begin{cases}\dfrac{1}{64}\left(10-xy^2\right), & 2\le x\le 10, 0\le y\le 1\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the expected age of an insured automobile involved in an accident.

(A) 4.9
(B) 5.2
(C) 5.8
(D) 6.0
(E) 6.4

121. Solution: C

The marginal density of X is given by

$$f_X(x) = \int_0^1 \frac{1}{64}\left(10 - xy^2\right)dy = \frac{1}{64}\left(10y - \frac{xy^3}{3}\right)\Bigg|_0^1 = \frac{1}{64}\left(10 - \frac{x}{3}\right)$$

Then,

$$E[X] = \int_2^{10} x f_X(x)dx = \int_2^{10} x\frac{1}{64}\left(10 - \frac{x}{3}\right)dx = \frac{1}{64}\left(\frac{10x^2}{2} - \frac{x^3}{9}\right)\Bigg|_2^{10}$$

$$= \frac{1}{64}\left(\frac{1000}{2} - \frac{1000}{9} - \frac{40}{2} + \frac{8}{9}\right) = 5.78.$$

125. The distribution of Y, given X, is uniform on the interval $[0, X]$. The marginal density of X is

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the conditional density of X, given $Y = y$ where positive.

(A) 1
(B) 2
(C) $2x$
(D) $1/y$
(E) $1/(1-y)$

125. Solution: E

The joint density is $f(x, y) = f_X(x) f(y \mid x) = 2x(1/x) = 2, 0 < y < x < 1$.

The marginal density of Y is $f_Y(y) = \int_y^1 2dx = 2(1-y)$.

The conditional density is $f(x \mid y) = \dfrac{f(x, y)}{f_Y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}$.

144. A client spends X minutes in an insurance agent's waiting room and Y minutes meeting with the agent. The joint density function of X and Y can be modeled by

$$f(x,y)=\begin{cases}\frac{1}{800}e^{-\frac{x}{40}}e^{-\frac{y}{20}}, & x>0,\ y>0\\ 0, & \text{otherwise.}\end{cases}$$

Determine which of the following expressions represents the probability that a client spends less than 60 minutes at the agent's office.

(A) $\frac{1}{800}\int_0^{40}\int_0^{20} e^{-\frac{x}{40}}e^{-\frac{y}{20}}dydx$

(B) $\frac{1}{800}\int_0^{40}\int_0^{20-x} e^{-\frac{x}{40}}e^{-\frac{y}{20}}dydx$

(C) $\frac{1}{800}\int_0^{20}\int_0^{40-x} e^{-\frac{x}{40}}e^{-\frac{y}{20}}dydx$

(D) $\frac{1}{800}\int_0^{60}\int_0^{60} e^{-\frac{x}{40}}e^{-\frac{y}{20}}dydx$

(E) $\frac{1}{800}\int_0^{60}\int_0^{60-x} e^{-\frac{x}{40}}e^{-\frac{y}{20}}dydx$

144. Solution: E

The total time is to e less than 60 minutes, so if x minutes are spent in the waiting room (in the range 0 to 60), from 0 to $60 - x$ minutes are spent in the meeting itself.

162. The joint probability density function of X and Y is given by

$$f(x,y)=\begin{cases}\dfrac{x+y}{8}, & 0<x<2, 0<y<2\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the variance of $(X+Y)/2$.

(A) 10/72
(B) 11/72
(C) 12/72
(D) 20/72
(E) 22/72

162. Solution: A

First, observe that

$Var[(X+Y)/2] = (0.5)^2 Var(X+Y) = 0.25[Var(X)+Var(Y)+2Cov(X,Y)]$.

Then,

$$E(X) = \int_0^2\int_0^2 x\frac{x+y}{8}dxdy = \frac{1}{8}\int_0^2 \frac{8}{3}+2ydy = \frac{1}{8}\left(\frac{16}{3}+4\right) = \frac{2}{3}+\frac{1}{2} = \frac{7}{6}$$

$$E(X^2) = \int_0^2\int_0^2 x^2\frac{x+y}{8}dxdy = \frac{1}{8}\int_0^2 4+\frac{8}{3}ydy = \frac{1}{8}\left(8+\frac{16}{3}\right) = 1+\frac{4}{6} = \frac{10}{6}$$

$Var(X) = 10/6-(7/6)^2 = 11/36.$

By symmetry, the mean and the variance of Y are the same. Next,

$$E(XY) = \int_0^2\int_0^2 xy\frac{x+y}{8}dxdy = \frac{1}{8}\int_0^2 \frac{8}{3}y+2y^2dy = \frac{1}{8}\left(\frac{16}{3}+\frac{16}{3}\right) = \frac{8}{6},$$

$Cov(X,Y) = 8/6-(7/6)(7/6) = -1/36.$

Finally,

$Var(X+Y) = 0.25[11/36+11/36+2(-1/36)] = 5/36 = 10/72.$

203. A machine has two components and fails when both components fail. The number of years from now until the first component fails, X, and the number of years from now until the machine fails, Y, are random variables with joint density function

$$f(x,y)=\begin{cases}\frac{1}{18}e^{-(x+y)/6}, & 0<x<y\\ 0, & \text{otherwise.}\end{cases}$$

H.W

Calculate $\mathrm{Var}(Y \mid X=2)$.

(A) 6
(B) 8
(C) 36
(D) 45
(E) 64

203. Solution: C

The conditional density of Y given $X = 2$ is

$$f_{Y|X}(y\,|\,2) = \frac{f_{X,Y}(2,y)}{f_X(2)} = \frac{\frac{1}{18}e^{-(2+y)/6}}{\int_2^\infty \frac{1}{18}e^{-(2+y)/6}dy} = \frac{\frac{1}{18}e^{-(2+y)/6}}{-\frac{1}{3}e^{-(2+y)/6}\Big|_2^\infty} = \frac{\frac{1}{18}e^{-(2+y)/6}}{\frac{1}{3}e^{-2/3}} = \frac{1}{6}e^{-(y-2)/6}, \quad y > 2 \text{, and is}$$

zero otherwise.

While the mean and then the variance can be obtained from the usual integrals, it is more efficient to recognize that this density function is 2 more than an exponential random variable with mean 6. The variance is then the same as that for an exponential random variable with mean 6, which is 6x6 = 36.

204. The elapsed time, T, between the occurrence and the reporting of an accident has probability density function

$$f(t) = \begin{cases} \dfrac{8t - t^2}{72}, & 0 < t < 6 \\ 0, & \text{otherwise.} \end{cases}$$

H.W

Given that $T = t$, the elapsed time between the reporting of the accident and payment by the insurer is uniformly distributed on $[2 + t, 10]$.

Calculate the probability that the elapsed time between the occurrence of the accident and payment by the insurer is less than 4.

(A) 0.005
(B) 0.023
(C) 0.033
(D) 0.035
(E) 0.133

204. Solution: A

Let Y denote the time between report and payment. Then

$$f(t,y) = f(y\,|\,t)f(t) = \left(\frac{1}{8-t}\right)\left(\frac{8t-t^2}{72}\right) = \frac{t}{72}, \quad 0 < t < 6, 2+t < y < 10$$

$$\mathrm{P}(T+Y<4) = \int_0^1 \int_{2+t}^{4-t} \frac{t}{72} dy dt = \int_0^1 t \frac{(4-t)-(2+t)}{72} dt = \int_0^1 t \frac{2-2t}{72} dt = \left. \frac{t^2 - 2t^3/3}{72} \right|_0^1 = 1/216 = 0.005.$$

228. As a block of concrete is put under increasing pressure, engineers measure the pressure X at which the first fracture appears and the pressure Y at which the second fracture appears. X and Y are measured in tons per square inch and have joint density function

$$f(x,y) = \begin{cases} 24x(1-y), & 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

H.W

Calculate the average pressure (in tons per square inch) at which the second fracture appears, given that the first fracture appears at 1/3 ton per square inch.

(A) 4/9
(B) 5/9
(C) 2/3
(D) 3/4
(E) 80/81

228. Solution: B

The marginal density of X at 1/3 is $\int_{1/3}^{1} 24(1/3)(1-y)dy = 16/9$. The conditional density of Y given $X = 1/3$ is $\dfrac{24(1/3)(1-y)}{16/9} = 4.5(1-y), 1/3 < y < 1$. The mean is

$$4.5\int_{1/3}^{1} y(1-y)dy = 2.25y^2 - 1.5y^3\Big|_{1/3}^{1} = 5/9.$$

230. Let X denote the proportion of employees at a large firm who will choose to be covered under the firm's medical plan, and let Y denote the proportion who will choose to be covered under both the firm's medical and dental plans.

Suppose that for $0 \le y \le x \le 1$, X and Y have the joint cumulative distribution function

$$F(x, y) = y(x^2 + xy - y^2).$$

Calculate the expected proportion of employees who will choose to be covered under both plans.

H.W

(A) 0.06
(B) 0.33
(C) 0.42
(D) 0.50
(E) 0.75

230. Solution: C

$$F_Y(y) = F(1, y) = y + y^2 - y^3 \Rightarrow f_Y(y) = 1 + 2y - 3y^2$$

$$E(Y) = \int_0^1 y(1 + 2y - 3y^2)dy = 1/2 + 2(1/3) - 3(1/4) = 5/12 = 0.417.$$

232. An insurance company sells automobile liability and collision insurance. Let X denote the percentage of liability policies that will be renewed at the end of their terms and Y the percentage of collision policies that will be renewed at the end of their terms. X and Y have the joint cumulative distribution function

$$F(x,y)=\frac{xy(x+y)}{2,000,000},\ 0\le x\le 100,\ 0\le y\le 100.$$

H.W

Calculate Var(X).

(A) 764
(B) 833
(C) 3402
(D) 4108
(E) 4167

232. Solution: A

$$F_X(x) = F(x,100) = \frac{100x(x+100)}{2,000,000} = \frac{100x^2 + 10,000x}{2,000,000} \Rightarrow f_X(x) = \frac{x}{10,000} + \frac{1}{200}$$

$$E(X) = \int_0^{100} \frac{x^2}{10,000} + \frac{x}{200}\,dx = \left.\frac{x^3}{30,000} + \frac{x^2}{400}\right|_0^{100} = 58.33$$

$$E(X^2) = \int_0^{100} \frac{x^3}{10,000} + \frac{x^2}{200}\,dx = \left.\frac{x^4}{40,000} + \frac{x^3}{600}\right|_0^{100} = 4166.67$$

$$Var(X) = 4166.67 - 58.33^2 = 764.$$

233. A hurricane policy covers both water damage, X, and wind damage, Y, where X and Y have joint density function

$$f(x,y)=\begin{cases}0.13e^{-0.5x-0.2y}-0.06e^{-x-0.2y}-0.06e^{-0.5x-0.4y}+0.12e^{-x-0.4y}, & x>0, y>0\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the standard deviation of X.

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

H.W

233. Solution: B

The marginal distribution is

$f_X(x) = \int_0^\infty f(x,y)dy = 0.65e^{-0.5x} - 0.30e^{-x} - 0.15e^{-0.5x} + 0.30e^{-x} = 0.5e^{-0.5x}$. This is an exponential distribution with a mean of 1/0.5 = 2. The standard deviation is equal to the mean.