جامعة الملك سعود
King Saud University

STAT415
probability(2)

part4 –problems from SOA

29. The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year.

Calculate the portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year.

(A) 0.15
(B) 0.34
(C) 0.43
(D) 0.57
(E) 0.66

29. Solution: C

Let T denote the number of days that elapse before a high-risk driver is involved in an accident. Then T is exponentially distributed with unknown parameter λ. We are given that

$$0.3 = P[T \le 50] = \int_0^{50} \lambda e^{-\lambda t} dt = -e^{-\lambda t} \Big|_0^{50} = 1 - e^{-50\lambda}.$$

Therefore, $e^{=50\lambda} = 0.7$ and $\lambda = -(1/50)\ln(0.7)$.

Then,

$$P[T \le 80] = \int_0^{80} \lambda e^{-\lambda t} dt = -e^{-\lambda t} \Big|_0^{80} = 1 - e^{-80\lambda} \quad = 1 - e^{-80\lambda}$$

$$= 1 - e^{(80/50)\ln(0.7)} = 1 - 0.7^{8/5} = 0.435.$$

33. The loss due to a fire in a commercial building is modeled by a random variable X with density function

$$f(x) = \begin{cases} 0.005(20 - x), & 0 < x < 20 \\ 0, & \text{otherwise.} \end{cases}$$

Given that a fire loss exceeds 8, calculate the probability that it exceeds 16.

(A) 1/25
(B) 1/9
(C) 1/8
(D) 1/3
(E) 3/7

33. Solution: B

$$P[X > x] = \int_x^{20} 0.005(20 - t)\,dt = 0.005\left(20t - \frac{1}{2}t^2\right)\Big|_x^{20}$$

$$= 0.005\left(400 - 200 - 20x + \frac{1}{2}x^2\right) = 0.005\left(200 - 20x + \frac{1}{2}x^2\right)$$

where $0 < x < 20$. Therefore,

$$P\left[X > 16 \middle| X > 8\right] = \frac{P[X > 16]}{P[X > 8]} = \frac{200 - 20(16) + \frac{1}{2}(16)^2}{200 - 20(8) + \frac{1}{2}(8)^2} = \frac{8}{72} = \frac{1}{9}.$$

34. The lifetime of a machine part has a continuous distribution on the interval (0, 40) with probability density function $f(x)$, where $f(x)$ is proportional to $(10 + x)^{-2}$ on the interval.

Calculate the probability that the lifetime of the machine part is less than 6.

(A) 0.04
(B) 0.15
(C) 0.47
(D) 0.53
(E) 0.94

34. Solution: C

We know the density has the form $C(10+x)^{-2}$ for $0 < x < 40$ (equals zero otherwise). First, determine the proportionality constant C.

$$1 = \int_0^{40} C(10+x)^{-2}\,dx = -C(10+x)^{-1}\Big|_0^{40} = \frac{C}{10} - \frac{C}{50} = \frac{2}{25}C$$

So $C = 25/2$ or 12.5. Then, calculate the probability over the interval (0, 6):

$$12.5\int_0^{6} (10+x)^{-2}\,dx = -12.5(10+x)^{-1}\Big|_0^{6} = 12.5\left(\frac{1}{10} - \frac{1}{16}\right) = 0.47.$$

36. A group insurance policy covers the medical claims of the employees of a small company. The value, V, of the claims made in one year is described by

$$V = 100{,}000Y$$

where Y is a random variable with density function

$$f(y) = \begin{cases} k(1-y)^4, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

Calculate the conditional probability that V exceeds 40,000, given that V exceeds 10,000.

(A) 0.08
(B) 0.13
(C) 0.17
(D) 0.20
(E) 0.51

36. Solution: B

To determine *k*,

$$1 = 1 = \int_0^1 k(1-y)^4\,dy = -\frac{k}{5}(1-y)^5\Big|_0^1 = \frac{k}{5}\text{, so } k = 5$$

We next need to find P[V > 10,000] = P[100,000 Y > 10,000] = P[Y > 0.1], which is

$\int_{0.1}^1 5(1-y)^4 dy = -(1-y)^5\Big|_{0.1}^1 = 0.9^5 = 0.59$ and P[V > 40,000] which is

$\int_{0.4}^1 5(1-y)^4 dy = -(1-y)^5\Big|_{0.4}^1 = 0.6^5 = 0.078$. Then,

$$P[V > 40,000 \mid V > 10,000] = \frac{P[V > 40,000 \cap V > 10,000]}{P[V > 10,000]} = \frac{P[V > 40,000]}{P[V > 10,000]} = \frac{0.078}{0.590} = 0.132.$$

37. The lifetime of a printer costing 200 is exponentially distributed with mean 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, a one-half refund if it fails during the second year, and no refund for failure after the second year.

Calculate the expected total amount of refunds from the sale of 100 printers.

(A) 6,321
(B) 7,358
(C) 7,869
(D) 10,256
(E) 12,642

37. Solution: D

Let T denote printer lifetime. The distribution function is $F(t) = 1 - e^{-t/2}$. The probability of failure in the first year is $F(1) = 0.3935$ and the probability of failure in the second year is $F(2) - F(1) = 0.6321 - 0.3935 = 0.2386$. Of 100 printers, the expected number of failures is 39.35 and 23.86 for the two periods. The total expected cost is 200(39.35) + 100(23.86) = 10,256.

38. An insurance company insures a large number of homes. The insured value, X, of a randomly selected home is assumed to follow a distribution with density function

$$f(x) = \begin{cases} 3x^{-4}, & x > 1 \\ 0, & \text{otherwise.} \end{cases}$$

Given that a randomly selected home is insured for at least 1.5, calcuate the probability that it is insured for less than 2.

(A) 0.578
(B) 0.684
(C) 0.704
(D) 0.829
(E) 0.875

38. Solution: A

The distribution function is $F(x) = P[X \le x] = \int_1^x 3t^{-4}dt = -t^{-3}\Big|_1^x = 1 - x^{-3}$. Then,

$$P[X < 2 \mid X \ge 1.5] = \frac{P\big[(X < 2) \text{ and } (X \ge 1.5)\big]}{P\big[X \ge 1.5\big]} = \frac{P[X < 2] - \Pr[X < 1.5]}{\Pr\big[X \ge 1.5\big]}$$

$$= \frac{F(2) - F(1.5)}{1 - F(1.5)} = \frac{(1-2^{-3}) - (1-1.5^{-3})}{1-(1-1.5^{-3})} = \frac{-1/8 + 8/27}{8/27} = \frac{37}{64} = 0.578$$

40. An insurance policy pays for a random loss X subject to a deductible of C, where $0 < C < 1$. The loss amount is modeled as a continuous random variable with density function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Given a random loss X, the probability that the insurance payment is less than 0.5 is equal to 0.64.

Calculate C.

(A) 0.1
(B) 0.3
(C) 0.4
(D) 0.6
(E) 0.8

40. Solution: B

Denote the insurance payment by the random variable Y. Then

$$Y = \begin{cases} 0 & \text{if} \quad 0 < X \le C \\ X - C & \text{if} \quad \mathrm{C} < X < 1 \end{cases}$$

We are given that

$$0.64 = P[Y < 0.5] = P[0 < X < 0.5 + C] = \int_0^{0.5+C} 2x\,dx = x^2 \Big|_0^{0.5+C} = (0.5 + C)^2 .$$

The quadratic equation has roots at C = 0.3 and 1.3. Because C must be less than 1, the solution is $C = 0.3$.

45. Let X be a continuous random variable with density function

$$f(x)=\begin{cases}\dfrac{|x|}{10}, & -2\le x\le 4\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the expected value of X.

(A) 1/5
(B) 3/5
(C) 1
(D) 28/15
(E) 12/5

45. Solution: D

$$E(X)=\int_{-2}^{0}x\frac{-x}{10}dx+\int_{0}^{4}x\frac{x}{10}dx=-\frac{x^3}{30}\bigg|_{-2}^{0}+\frac{x^3}{30}\bigg|_{0}^{4}=-\frac{8}{30}+\frac{64}{30}=\frac{56}{30}=\frac{28}{15}$$

55. An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1 + x)^{-4}$, for $0 < x < \infty$. Calculate the company's expected monthly claims.

(A) 1/6
(B) 1/3
(C) 1/2
(D) 1
(E) 3

55. Solution: C

$$1 = \int_0^\infty \frac{k}{(1+x)^4}\,dx = -\frac{k}{3}\frac{1}{(1+x)^3}\Bigg|_0^\infty = \frac{k}{3} \text{ and so } k = 3.$$

The expected value is (where the substitution $u = 1 + x$ is used.

$$\int_0^\infty x\frac{3}{(1+x)^4}\,dx = \int_1^\infty 3(u-1)u^{-4}du = 3u^{-2}/(-2) - 3u^{-3}/(-3)\Big|_1^\infty = 3/2 - 1 = 1/2.$$

Integration by parts may also be used.

Book : chapter3 page 92

The **(100p)th percentile** is a number π_p such that the area under $f(x)$ to the left of π_p is p. That is,

$$p = \int_{-\infty}^{\pi_p} f(x)\,dx = F(\pi_p).$$

The 50th percentile is called the **median**. We let $m = \pi_{0.50}$. The 25th and 75th percentiles are called the **first** and **third quartiles**, respectively, and are denoted by $q_1 = \pi_{0.25}$ and $q_3 = \pi_{0.75}$. Of course, the median $m = \pi_{0.50} = q_2$ is also called the **second quartile**.

59. An insurer's annual weather-related loss, X, is a random variable with density function

$$f(x) = \begin{cases} \dfrac{2.5(200)^{2.5}}{x^{3.5}}, & x > 200 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the difference between the 30th and 70th percentiles of X.

(A) 35
(B) 93
(C) 124
(D) 231
(E) 298

59. Solution: B

The distribution function of X is

$$F(x) = \int_{200}^{x} \frac{2.5(200)^{2.5}}{t^{3.5}}\,dt = \left.\frac{-(200)^{2.5}}{t^{2.5}}\right|_{200}^{x} = 1 - \frac{(200)^{2.5}}{x^{2.5}}, \quad x > 200$$

The pth percentile x_p of X is given by

$$\frac{p}{100} = F(x_p) = 1 - \frac{(200)^{2.5}}{x_p^{2.5}}$$

$$1 - 0.01p = \frac{(200)^{2.5}}{x_p^{2.5}}$$

$$(1 - 0.01p)^{0.4} = \frac{200}{x_p}$$

$$x_p = \frac{200}{(1 - 0.01p)^{0.4}}$$

It follows that $x_{70} - x_{30} = \dfrac{200}{(0.30)^{0.4}} - \dfrac{200}{(0.70)^{0.4}} = 93.06$.

62. A random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 1 \\ \dfrac{x^2 - 2x + 2}{2}, & 1 \le x < 2 \\ 1, & x \ge 2. \end{cases}$$

Calculate the variance of X.

(A) 7/72
(B) 1/8
(C) 5/36
(D) 4/3
(E) 23/12

62. Solution: C

First note that the distribution function jumps ½ at $x = 1$, so there is discrete probability at that point. From 1 to 2, the density function is the derivative of the distribution function, $x - 1$. Then,

$$E(X) = \frac{1}{2}(1) + \int_1^2 x(x-1)dx = \frac{1}{2} + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_1^2 = \frac{1}{2} + \frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2} = \frac{4}{3}$$

$$E(X^2) = \frac{1}{2}(1)^2 + \int_1^2 x^2(x-1)dx = \frac{1}{2} + \left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Bigg|_1^2 = \frac{1}{2} + \frac{16}{4} - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} = \frac{23}{12}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{23}{12} - \left(\frac{4}{3}\right)^2 = \frac{23}{12} - \frac{16}{9} = \frac{5}{36}.$$

63. The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X, has density function

$$f(x) = \begin{cases} 1/5, & 0 < x < 5 \\ 0, & \text{otherwise.} \end{cases}$$

Let Y be the age of the machine at the time of replacement.

Calculate the variance of Y.

(A) 1.3
(B) 1.4
(C) 1.7
(D) 2.1
(E) 7.5

63. Solution: C

$$E[Y] = \int_0^4 x(0.2)dx + \int_4^5 4(0.2)dx = 0.1x^2\Big|_0^4 + 0.8 = 2.4$$

$$E[Y^2] = \int_0^4 x^2(0.2)dx + \int_4^5 4^2(0.2)dx = (0.2/3)x^3\Big|_0^4 + 3.2 = 7.46667$$

$$\text{Var[Y]} = E[Y^2] - \left(E[Y]\right)^2 = 7.46667 - 2.4^2 = 1.707.$$

$$\mathsf{E}(\min(X,k)) = \int_0^\infty \min(x,k)\, f(x)\, \mathrm{d}x$$

$$= \int_0^k x\, f(x)\, \mathrm{d}x + k\int_k^\infty f(x)\mathrm{d}x$$

68. An insurance policy reimburses dental expense, X, up to a maximum benefit of 250. The probability density function for X is:

$$f(x) = \begin{cases} ce^{-0.004x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where c is a constant.

Calculate the median benefit for this policy.

(A) 161
(B) 165
(C) 173
(D) 182
(E) 250

68. Solution: C

X has an exponential distribution. Therefore, $c = 0.004$ and the distribution function is $F(x) = 1 - e^{-0.004x}$. For the moment, ignore the maximum benefit. The median is the solution to $0.5 = F(m) = 1 - e^{-0.004m}$, which is $m = -250\ln(0.5) = 173.29$. Because this is below the maximum benefit, it is the median regardless of the existence of the maximum. Note that had the question asked for a percentile such that the solution without the maximum exceeds 250, then the answer is 250.

69. The time to failure of a component in an electronic device has an exponential distribution with a median of four hours.

Calculate the probability that the component will work without failing for at least five hours.

(A) 0.07
(B) 0.29
(C) 0.38
(D) 0.42
(E) 0.57

69. Solution: D

The distribution function of an exponential random variable, T, is $F(t) = 1 - e^{-t/\theta}$, $t > 0$. With a median of four hours, $0.5 = F(4) = 1 - e^{-4/\theta}$ and so $\theta = -4 / \ln(0.5)$. The probability the component works for at least five hours is $P[T \geq 5] = 1 - F(5) = 1 - 1 + e^{5\ln(0.5)/4} = 0.5^{5/4} = 0.42$.

157. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \dfrac{p-1}{x^p}, & x > 1 \\ 0, & \text{otherwise} \end{cases}$$

Calculate the value of p such that $E(X) = 2$.

(A) 1
(B) 2.5
(C) 3
(D) 5
(E) There is no such p.

157. Solution: C

$$E(X) = \int_1^\infty x \frac{p-1}{x^p} dx = (p-1)\int_1^\infty x^{1-p} dx$$

$$(p-1)\frac{x^{2-p}}{2-p}\Bigg|_1^\infty = \frac{p-1}{p-2} = 2$$

$$p-1 = 2(p-2) = 2p-4$$

$$p = 3$$

178. The proportion X of yearly dental claims that exceed 200 is a random variable with probability density function

$$f(x) = \begin{cases} 60x^3(1-x)^2, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate Var[X/(1 – X)]

(A) 149/900
(B) 10/7
(C) 6
(D) 8
(E) 10

178. Solution: C

$$E\left(\frac{X}{1-X}\right)=60\int_0^1 \frac{x}{1-x}x^3(1-x)^2dx=60\int_0^1 x^4(1-x)dx=60(x^5/5-x^6/6)\Big|_0^1=60(1/5-1/6)=2$$

$$E\left[\left(\frac{X}{1-X}\right)^2\right]=60\int_0^1 \frac{x^2}{(1-x)^2}x^3(1-x)^2dx=60\int_0^1 x^5dx=60(x^6/6)\Big|_0^1=60(1/6)=10$$

$$Var\left(\frac{X}{1-X}\right)=10-2^2=6$$

193. The lifespan, in years, of a certain computer is exponentially distributed. The probability that its lifespan exceeds four years is 0.30.

Let $f(x)$ represent the density function of the computer's lifespan, in years, for $x > 0$.

Determine which of the following is an expression for $f(x)$.

(A) $1-(0.3)^{-x/4}$

(B) $1-(0.7)^{x/4}$

(C) $1-(0.3)^{x/4}$

(D) $-\frac{\ln 0.7}{4}(0.7)^{x/4}$

(E) $-\frac{\ln 0.3}{4}(0.3)^{x/4}$

193. Solution: E

The cumulative distribution function for the exponential distribution of the lifespan is
$F(x) = 1 - e^{-\lambda x}$, for positive x.

The probability that the lifespan exceeds 4 years is $0.3 = 1 - F(4) = e^{-4\lambda}$. Thus $\lambda = -(\ln 0.3)/4$.
For positive x, the probability density function is

$$f(x) = \lambda e^{-\lambda x} = -\frac{\ln 0.3}{4} \mathrm{e}^{(\ln 0.3)x/4} = -\frac{\ln 0.3}{4}(0.3)^{x/4}.$$

194. The lifetime of a light bulb has density function, f, where $f(x)$ is proportional to

$$\frac{x^2}{1+x^3}, \quad 0 < x < 5 \text{, and 0, otherwise.}$$

Calculate the mode of this distribution.

(A) 0.00
(B) 0.79
(C) 1.26
(D) 4.42
(E) 5.00

194. Solution: C

It is not necessary to determine the constant of proportionality. Let it be c. To determine the mode, set the derivative of the density function equal to zero and solve.

$$0 = f'(x) = \frac{d}{dx} cx^2(1+x^3)^{-1} = 2cx(1+x^3)^{-1} + cx^2[-(1+x^3)^{-2}]3x^2$$

$$= 2cx(1+x^3) - 3cx^4 \quad \text{(multiplying by } (1+x^3)^2\text{)}$$

$$= 2cx + 2cx^4 - 3cx^4 = 2cx - cx^4$$

$$= 2 - x^3 \Rightarrow x = 2^{1/3} = 1.26.$$

195. An insurer's medical reimbursements have density function f, where $f(x)$ is proportional to

xe^{-x^2}, for $0 < x < 1$, and 0, otherwise.

Calculate the mode of the medical reimbursements.

(A) 0.00
(B) 0.50
(C) 0.71
(D) 0.84
(E) 1.00

195. Solution: C

It is not necessary to determine the constant of proportionality. Let it be *c*. To determine the mode, set the derivative of the density function equal to zero and solve.

$$0 = f'(x) = \frac{d}{dx} cxe^{-x^2} = ce^{-x^2} - cx(2x)e^{-x^2} = ce^{-x^2}(1-2x^2)$$

$= 1 - 2x^2$ (multiplying by ce^{x^2})

$\Rightarrow x = (1/2)^{1/2} = 0.71.$

215. The distribution of the size of claims paid under an insurance policy has probability density function

$$f(x) = \begin{cases} cx^a, & 0 < x < 5 \\ 0, & \text{otherwise,} \end{cases}$$

Where $a > 0$ and $c > 0$.

For a randomly selected claim, the probability that the size of the claim is less than 3.75 is 0.4871.

Calculate the probability that the size of a randomly selected claim is greater than 4.

(A) 0.404
(B) 0.428
(C) 0.500
(D) 0.572
(E) 0.596

215. Solution: B

Because the density function must integrate to 1, $1 = \int_0^5 cx^a dx = c\dfrac{5^{a+1}}{a+1} \Rightarrow c = \dfrac{a+1}{5^{a+1}}$.

From the given probability,

$$0.4871 = \int_0^{3.75} cx^a dx = c\frac{3.75^{a+1}}{a+1} = \frac{a+1}{5^{a+1}}\frac{3.75^{a+1}}{a+1} = \left(\frac{3.75}{5}\right)^{a+1}$$

$$\ln(0.4871) = -0.71929 = (a+1)\ln(3.75/5) = -0.28768(a+1)$$

$$a = (-0.71929)/(-0.28768) - 1 = 1.5.$$

The probability of a claim exceeding 4 is,

$$\int_4^5 cx^a dx = c\frac{5^{a+1} - 4^{a+1}}{a+1} = \frac{a+1}{5^{a+1}}\frac{5^{a+1} - 4^{a+1}}{a+1} = 1 - \left(\frac{4}{5}\right)^{1.5+1} = 0.42757.$$

190. A certain brand of refrigerator has a useful life that is normally distributed with mean 10 years and standard deviation 3 years. The useful lives of these refrigerators are independent.

Calculate the probability that the total useful life of two randomly selected refrigerators will exceed 1.9 times the useful life of a third randomly selected refrigerator.

(A) 0.407
(B) 0.444
(C) 0.556
(D) 0.593
(E) 0.604

H.W

190. Solution: C

Let X, Y, and Z be the three lifetimes. We want
$P(X+Y>1.9Z)=P(W=X+Y-1.9Z>0)$.
A linear combination of independent normal variables is also normal. In this case W has a mean of 10 + 10 – 1.9(10) = 1 and a variance of 9 + 9 + 1.9(1.9)(9) = 50.49 for a standard deviation of 7.106.
Then the desired probability is that a standard normal variable exceeds (0 – 1)/7.106 = –0.141.
Interpolating in the normal tables gives 0.5557 + (0.5596 – 0.5557)(0.1) = 0.5561, which rounds to 0.556.

175. An insurance company's annual profit is normally distributed with mean 100 and variance 400.

Let Z be normally distributed with mean 0 and variance 1 and let F be the cumulative distribution function of Z.

Determine the probability that the company's profit in a year is at most 60, given that the profit in the year is positive.

(A) $1 - F(2)$

(B) $F(2)/F(5)$

(C) $[1 - F(2)]/F(5)$

(D) $[F(0.25) - F(0.1)]/F(0.25)$

(E) $[F(5) - F(2)]/F(5)$

H.W

175. Solution: E

The profit variable X is normal with mean 100 and standard deviation 20. Then,

$$P(X \le 60 \mid X > 0) = \frac{P(0 < X \le 60)}{P(X > 0)} = \frac{P\left(\frac{0-100}{20} < Z \le \frac{60-100}{20}\right)}{P\left(Z > \frac{0-100}{20}\right)} = \frac{F(-2) - F(-5)}{1 - F(-5)}.$$

For the normal distribution, $F(-x) = 1 - F(x)$ and so the answer can be rewritten as $[1 - F(2) - 1 + F(5)]/[1 - 1 + F(5)] = [F(5) - F(2)]/F(5)$.