Sec. 3.5 --- Variations of the Sign Test

• Tests based on the sign test can be used to answer a variety of questions.

McNemar's Test

• Consider two paired binary variables (nominal).

• Both X and Y can only take the values 0 and 1, say.

• This type of data often arises from "before vs. after" experiments.

• $X = 1$ might represent having some condition before
a treatment is applied and $Y = 1$ having it after the treatment is applied.

• **Question**:

Is the probability of having the condition the same before and after the treatment is applied ?

• Or does the treatment change the probability of having it ?

Null and Alternative hypotheses:

$H_0: P(X_i=1) = P(Y_i=1)$ (for all i) vs. $H_1: P(X_i=1) \neq P(Y_i=1)$ (for all i)

$\Longleftrightarrow$

$H_0: P(X_i=1, Y_i=1) + P(X_i=1, Y_i=0) = P(X_i=1, Y_i=1) + P(X_i=0, Y_i=1)$

$\Longleftrightarrow$ $H_0: P(X_i=1, Y_i=0) = P(X_i=0, Y_i=1)$

• The data from such a study can be summarized with a 2 × 2 table:

		Y_i = 0	Y_i = 1
X_i	0	a	b
	1	c	d

a, b, c, d are the counts of observations falling in each cell

- Consider the ($X_i = 0$, $Y_i = 1$) entries to be the "+" observations.
- Let the ($X_i = 1$, $Y_i = 0$) entries be the "–" observations.

- Then we can use the sign test to test H_0.

- Note the *a* and *d* entries are treated as ties . **[a=(0,0) , b =(1,1)]**
- The test statistic is simply $T_2 = b$ (The number of +'s)

- The null distribution of T_2 is binomial (n =b + c , p =0.5)

Using Table A3 with $n = b + c$ and $p = 0.5$,

reject H_0 if

$T_2 \leq t$ or $T_2 \geq n-t$

where t is the value corresponding to a probability of $\approx \alpha/2$.

The P-value is

P-value $= 2x[\min\{P(T_2 \leq t_2^{obs}) , P(T_2 \geq t_2^{obs})\}]$

Example 1:

Suppose 200 subjects were asked last month and again this month whether they approved of the president's job performance. 90 said "yes" both times; 90 said "no" both times; 12 said "yes" the first month and "no" the second, and 8 said "no" the first month and "yes" the second. At $\alpha = 0.05$, has the president's approval rating significantly changed?

Contingency Table:

Let (no = 0 , yes = 1)

X = first month.
Y = next month.

first month	Second month: no	Second month: yes
no	90	8
yes	12	90

X_i \ Y_i	0	1
0	90	8
1	12	90

Hypotheses:

$H_0 : P(X_i=1) = P(Y_i=1)$ for all i

$H_1 : P(X_i=1) \neq P(Y_i=1)$ for all i

Test Statistic:

T_2= number of " +"

$T_2 = 8$

Look at binomial table with n=20 , p =0.5

n	y	P=0.5
20	. .	
	5	0.0207
	6	0.0577

0.025

$P(Y \leq 5) = 0.0207$

Reject H_0 if

$T_2 \leq t$ or $T_2 \geq n - t = 15$

$T_2 \leq 5$ or $T_2 \geq 20 - 5 = 15$

Since

$8 \nleq 5$ or $8 \ngeq 15$ (not satisfies - Accept H_0)

P-value:

P-value $= 2x[\min\{P(T_2 \leq 8) , P(T_2 \geq 8)\}]$

$= 2x[\min\{P(T_2 \leq 8) , 1- P(T_2 \leq 7)\}]$

$= 2x[0.2517 , (1-0.1316)]$

$= 2x[0.2517 , 0.8684]$

$= 2 \times 0.2517$

$= 0.5034$

n	y	P=0.5
20	. .	
	7	0.1316
	8	0.2517

Conclusion:

At $\alpha = 0.05$, fail to reject H_0 .

We cannot conclude that the approval rating significantly changed.

R code:

```
datamatrix<-matrix(c(90,12,8,90),ncol=2)
mcnemar.test(datamatrix, correct=FALSE)

# This gives T1 and a chi-squared based p-value.
# Note that R uses the large sample approximation to get the p-value.

# For the exact test, you could also use the binomial test of 0.5 for this example,
# where the number of (0,1) is 8 and number of (1,0) is 12.

binom.test(8,8+12,0.5)
```

Output:

```
                    Exact binomial test

data:  8 and 8 + 12
number of successes = 8, number of trials = 20, p-value = 0.5034
alternative hypothesis: true probability of success is not equal to 0.5
95 percent confidence interval:
 0.1911901 0.6394574
sample estimates:
probability of success
            0.4
```

• **For large samples (n > 20),** we can use the test statistic

$$T_1 = \frac{(b-c)^2}{b+c}$$

which has a χ_1^2 null distribution (approximately, for large samples)

Why is this?

$T_2 = b$ is binomial(b+c ,0.5) under H_0

Then,

If (b+c) is large , we can approximate this with a normal distribution

With mean (0.5)(b+c) and standard deviation $\sqrt{(b+c)x0.5x0.5}$

So,

$$Z = \frac{T_2 - \frac{b+c}{2}}{\sqrt{(b+c)x0.5x0.5}} = \frac{b-0.5b-0.5c}{0.2\sqrt{b+c}} = \frac{0.5(b-c)}{0.2\sqrt{b+c}}$$

$Z = \frac{b-c}{\sqrt{b+c}}$ has an approximate N(0,1) distribution

⟶ $Z^2 = \frac{(b-c)^2}{b+c}$ has an approximate χ^2 distribution

(we can get the value from Table A2 with 1 degree of freedom at 1-α)

Cox-Stuart Test for Trend

• An ordered sequence of numbers exhibits trend if the later numbers in the sequence tend to be greater than the earlier numbers (increasing trend) or if the later numbers in the sequence tend to be less than the earlier numbers (decreasing trend).

• In the arranged data, we essentially pair points to the left of the middle ordered value with points to the right of the middle ordered value, and perform a sign test.

• We assume the data $X_1, X_2, \ldots, X_{n'}$ are at least ordinal in scale.

• Pair the data as $(X_1, X_{1+c}), (X_2, X_{2+c}), \ldots, (X_{n'-c}, X_{n'})$ where

$c = n'/2$ if n' is even;

$c = (n'+1)/2$ if n' is odd.

• **Note that if *n'* is odd**, the middle value is ignored.

• If the first element in a pair is **less than** the second, we write a "+" for that pair.

• If the first element in a pair is **greater than** the second, we write a "–" for that pair.

• If the first element in a pair **equals the second**, we ignore that pair.

Null hypothesis: H_0: no trend

3 possible alternatives:

Two-tailed	Lower-tailed	Upper-tailed
H_1:either increasing or decreasing trend	H_1: decreasing trend	H_1:either increasing

Test Statistic: T = total number of +'s

Null distribution of T is binomial with p = 0.5

and n = the number of untied pairs.

• The decision rule and p-value are obtained in the same way as the sign test.

Example:

NASA data give the average global temperatures for the last 13 decades, from the 1880s to the 2000s:

-0.493, -0.457, -0.466, -0.497, -0.315, -0.077, 0.063, -0.036, -0.025, -0.002, 0.317, 0.563, 0.923. (Temperatures given in degrees F, centered by subtracting from 1951-1980 mean).

Does Cox-Stuart test find evidence (at α = 0.05) of an increasing trend?

Hypotheses:

H_1:either increasing

$n' = 13$ and $c = \frac{13+1}{2} = 7$

Pairs:

(X_1,X_8), (X_2,X_9), (X_3,X_{10}), (X_4,X_{11}), (X_5,X_{12}), (X_6,X_{13})

(-0.493,-0.036),(-0.457,-0.025),(-0.466,-0.002),(-0.497,-0.317),(-0.315,0.563),(-0.077,0.923)

+ + + + + +

T = 6 (number of +'s)

Look at binomial table with p = 0.5 and n = 6 (number of pairs)

P-value = $P(T \geq T) = P(T \geq 6) = 1 - P(T \leq 5) = 1 - 0.9844 = 0.0156 < \alpha$

Conclusion:

We reject H_0. We can conclude that an increasing trend in global temperature.

• In order to test for a specified type of trend other than increasing or decreasing (such as periodic, alternating, etc.), the data must first be reordered to reflect the expected ordering according to the specified trend. Then the Cox-Stuart test can be implemented on the reordered data.

R code:

```
## Cox-Stuart test for trend:

global.temps <- c(-0.493, -0.457, -0.466, -0.497, -0.315, -0.077, 0.063, -0.036, -0.025, -0.002,
0.317, 0.563, 0.923)

binom.test(6,6,p=0.5,alternative = "greater")
```

Output:

```
        Exact binomial test

data:  6 and 6
number of successes = 6, number of trials = 6, p-value = 0.01563
alternative hypothesis: true probability of success is greater than 0.5
95 percent confidence interval:
 0.6069622 1.0000000
sample estimates:
probability of success
                     1
```