SOLUTION FINAL EXAM., SEMESTER I, 2024
DEPT. MATH., COLLEGE OF SCIENCE
KING SAUD UNIVERSITY
MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Q1. Considering the augmented matrix, we have

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & -2 & 3 & 1 \\ 1 & 2 & -1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -1 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so, the variable z is free. Hence, for $z = t \in \Re$, we get $y = \frac{1+4t}{4}$ and $x = \frac{3-2t}{2}$. [1+1+1=3]

(b) Since the determinant of A is 0, i.e., $det(A) = 0$, the system cannot be solve by Cramer's rule. [2]

(c) [1+1+2=4] Here $det(A) = 9$, the matrix C of the cofactors

$$= \begin{bmatrix} 1 & 4 & -2 \\ -5 & 7 & 1 \\ 3 & -6 & 3 \end{bmatrix}$$

so,

$$adj(A) = C^T = \begin{bmatrix} 1 & -5 & 3 \\ 4 & 7 & -6 \\ -2 & 1 & 3 \end{bmatrix}$$

Hence

$$A^{-1} = 1/9 \begin{bmatrix} 1 & -5 & 3 \\ 4 & 7 & -6 \\ -2 & 1 & 3 \end{bmatrix}$$

Q2. (a) $\mathbf{a} = \mathbf{AB} = \langle 1, -1, 3\rangle$, $\mathbf{b} = \mathbf{AC} = \langle 2, -3, 2\rangle$ and $\mathbf{c} = \mathbf{AD} = \langle 3, -4, 1\rangle$. Therefore, $V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = 4$ unit. [1+2=3]

(b) Given $\mathbf{r}(t) = \langle t\sin(2t); t\cos(2t)\rangle$. Then $\mathbf{v}(t) = \mathbf{r}'(t) = \langle \sin(2t) + 2t\cos(2t); \cos(2t) - 2t\sin(2t)\rangle$. Speed $\|\mathbf{v}(t)\| = \sqrt{1+4t^2}$. The acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 4\cos(2t) - 4t\sin(2t); -4\sin(2t) - 4t\cos(2t)\rangle$. At $t = 0$, $\mathbf{v}(0) = \langle 0, 1\rangle$, $\|\mathbf{v}(0)\| = 1$, and $\mathbf{a}(0) = \langle 4, 0\rangle$. [1+1+1=3]

Q3. (a) $\mathbf{r}(t) = \langle \cos t, \sin t, 1\rangle$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle$ and $\mathbf{r}''(t) = \langle -\cos t, -\sin t, 0\rangle$. Therefore, $\mathbf{a}_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|} = \frac{\cos t\sin t - \cos t\sin t}{\sqrt{\cos^2 t + \sin^2 t}} = 0$, $\mathbf{a}_N = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|} = 1$, and curvature, $\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = 1$. [1+1+1=3]

(b) [2] Domain of the function f,

$$\begin{aligned} D_f &= \{(x, y) \in \Re^2 | 1 + y - x \geq 0, x + y - 1 \geq 0, 1 - x^2 - y > 0\} \\ &= \{(x, y) \in \Re^2 | y \geq x - 1,\ y \geq -x + 1,\ y < 1 - x^2\} \end{aligned}$$

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Q3. (c) For the path $y = 0$, $x \to 0$, $\lim_{x\to 0} \frac{x^4}{x^4} = 1$
For the path $y = mx$, with $m \neq 0$, $x \to 0$,

$$\lim_{x\to 0} \frac{x^4+m^3x^3}{m^2x^2+(m^2x^2+x^2)^2} = \lim_{x\to 0} \frac{x^2+m^3x}{m^2+x^2(m^2+1)^2} = 0.$$

Hence, by two-path rule, limit does not exist. [1+2=3]
Q4. (a) $\frac{\partial p}{\partial r} = \frac{\partial p}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial p}{\partial w}\frac{\partial w}{\partial r} = 4u - 6v - 8w = 6r - 24s$
$\frac{\partial p}{\partial s} = \frac{\partial p}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial p}{\partial w}\frac{\partial w}{\partial s} = -2u + 2v - 8w = -14r - 2s.$ [1.5+1.5=3]
(b) $w = f(x^2 + y^2)$ with $u = x^2 + y^2$.
$\frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = \frac{dw}{du}(2x)$ and $\frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y} = \frac{dw}{du}(2y)$.
Therefore, $y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y} = 2xy\frac{dw}{du} - 2xy\frac{dw}{du} = 0$. [1+1=2]
(c) Given $f(x, y, z) = xy^2e^z$. The unit vector $\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle$.
Thus, $f_x(x, y, z) = y^2e^z$, $f_y(x, y, z) = 2xye^z$, and $f_z(x, y, z) = xy^2e^z$. We have

$$D_{\mathbf{u}}f(x, y, z) = \frac{2}{3}y^2e^z - \frac{2}{3}xye^z + \frac{2}{3}xy^2e^z.$$

Hence, at $P(2, -1, 0)$, we get $D_{\mathbf{u}}f(2, -1, 0) = \frac{10}{3}$. [1+1+1=3]
Q5. (a) $z = x^2 + y^2$ is a paraboloid. Consider $F(x, y, z) = x^2 + y^2 - z = 0$. Then $\nabla F = \langle 2x, 2y, -1 \rangle$. Therefore, $\nabla F(-1, 1, 2) = \langle -2, 2, -1 \rangle$. Thus, the equation of the tangent plane is $-2(x + 1) + 2(y - 1) - (z - 2) = 0$, i.e., $2x - 2y + z + 2 = 0$, and the parametric equations of the normal line are

$$x = -1 - 2t, \quad y = 1 + 2t, \quad z = 2 - t.$$

[1+1+1=3]

(b) [1+1+1+1=4] Given $f(x, y) = 9x^2y - 6x^3 + y^3 - 12y$. Then we have $f_x(x, y) = 18xy - 18x^2 = 18x(y - x)$ and $f_y(x, y) = 9x^2 + 3y^2 - 12$. Setting $f_x(x, y) = 0$ and $f_y(x, y) = 0$, and solving these equations, we get four critical points: $(0, 2)$, $(0, -2)$, $(1, 1)$, $(-1, -1)$.
Now $f_{xx}(x, y) = 18y - 36x$, $f_{yy}(x, y) = 6y$, and $f_{xy}(x, y) = 18x$. So, we have
$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2$.
For critical point $(0, 2)$, $D(0, 2) = 432 > 0$ and $f_{xx}(0, 2) = 36 > 0$ giving local minimum. For $(0, -2)$, $D(0, -2) = 432 > 0$ and $f_{xx}(0, -2) = -36 < 0$ giving local maximum. For other two critical points $(1, 1)$ and $(-1, -1)$, one obtains saddle points.

(c) [2] Let x, y, and z be three positive numbers. Considering $f(x, y, z) = xyz$ and $g(x, y, z) = x + y + z - 1 = 0$, we have $\nabla f = \lambda\nabla g$ which gives $\langle yz, xz, xy \rangle = \lambda\langle 1, 1, 1 \rangle$. Thus, we have

$$xyz = \lambda x,\ xyz = \lambda y,\ xyz = \lambda z \text{ and } x + y + z = 1.$$

Hence, we get $x = y = z = \frac{1}{3}$, and so, $x = \frac{1}{3}$, $y = \frac{1}{3}$ and $\frac{1}{3}$.