1-

If $f(x)$ is integrable on $[a,b]$, prove the inequality

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

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1) We know that:

$$-|f(x)| \le f(x) \le |f(x)| \quad \forall x,$$

and by monotonicity property:

$$\int_a^b -|f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

(Note that $|f(x)|$ is also integrable on $[a,b]$)

$$\Rightarrow -\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

2-

Prove that if $f(x)$ is a continuous function on $[a,b]$, then The integral $\int_a^x f(t)\,dt$, as a function of x, is uniformly continuous on $[a,b]$.

② Let $x, y \in [a,b]$, with $y \le x$.

Given $\epsilon > 0$ we look for $\delta > 0$ such that $x, y \in [a,b]$ such that $|x-y| < \delta \Rightarrow \left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right| < \epsilon$

$$\left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right| = \left| \int_a^x f(t)\,dt + \int_y^a f(t)\,dt \right|$$

$$= \left| \int_y^x f(t)\,dt \right| \quad \text{(Additive Property)}$$

$$\le \int_y^x |f(t)|\,dt \quad \text{(by Question 1)}$$

$$\le \int_y^x M\,dt \quad \text{(Since } |f| \text{ is continuous on } [a,b]\text{)} \quad M = \max_{a \le t \le b} |f(t)|$$

$$= M(x-y) < \epsilon.$$

$$\Rightarrow |x-y| < \frac{\epsilon}{M}$$

Take $\delta = \frac{\epsilon}{M}$

3-

a)

$$\lim_{n\to\infty} \frac{n^2}{n^4+1^4}+\frac{2n^2}{n^4+2^4}+\frac{3n^2}{n^4+3^4}+\cdots+\frac{n^3}{2n^4}$$

b)

$$\lim_{n\to\infty} \frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{n}{2n^2}$$

c)

$$\lim_{n\to\infty} \frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\frac{3^2}{n^3+3^3}+\cdots+\frac{n^2}{2n^3}$$

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3)

(a) $\lim_{n\to\infty}\left(\frac{n^2}{n^4+1^4}+\frac{2n^2}{n^4+2^4}+\dots+\frac{n\cdot n^2}{n^4+n^4}\right)$

$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{kn^2}{n^4+k^4} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{kn^2}{n^4(1+(\frac{k}{n})^4)} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})}{1+(\frac{k}{n})^4}$

where $f(x)=\frac{x^2}{1+x^4}$, and apply Questinon 5(1):

$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})}{1+(\frac{k}{n})^4} = \int_0^1 \frac{x}{1+x^4}dx = \frac{1}{2}\tan^{-1}(x^2)\Big]_0^1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$

(b) $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\dots+\frac{n}{n^2+n^2}\right)$

$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2(1+(\frac{k}{n})^2)} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+(\frac{k}{n})^2}$

Where $f(x)=\frac{1}{1+x^2}$, and apply Question 5(1):

$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+(\frac{k}{n})^2} = \int_0^1 \frac{1\cdot dx}{1+x^2} = \tan^{-1}(x)\Big]_0^1 = \frac{\pi}{4}$

(c) $\lim_{n\to\infty}\left(\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\frac{3^2}{n^3+3^3}+\dots+\frac{n^2}{n^3+n^3}\right)$

$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3+k^3} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3(1+(\frac{k}{n})^3)} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})^2}{1+(\frac{k}{n})^3}$

where $f(x)=\frac{x^2}{1+x^3}$, and apply Question 5(1):

$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(\frac{k}{n})^2}{1+(\frac{k}{n})^3} = \int_0^1 \frac{x^2\cdot dx}{1+x^3} = \frac{1}{2}\ln(1+x^3)\Big]_0^1 = \frac{1}{2}\ln(2)$

Q5:

1. Show that if $f\in\mathcal{R}(0,1)$, then $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx$.