Chapter 1

The Riemann Integral

Lecture 1: Partitions

Definition (Partition)

Let $[a, b]$ be a closed interval with $a < b$.
A **partition** of $[a, b]$ is a finite ordered set

$$P = \{x_0, x_1, \dots, x_n\} \quad \text{such that} \quad a = x_0 < x_1 < \cdots < x_n = b.$$

Subintervals

The partition P divides $[a, b]$ into the subintervals

$$[x_{k-1}, x_k], \qquad k = 1, 2, \dots, n.$$

These subintervals:

- are disjoint in their interiors,
- together cover the entire interval $[a, b]$.

A partition Q is called a **refinement** of a partition P if every point of P also belongs to Q.

A refinement is obtained by adding more points, making the partition finer.

Example

Consider the interval

$$[0, 6].$$

Let

$$P = \{0, 2, 5, 6\}.$$

This is a partition of $[0, 6]$, and it divides the interval into the subintervals

$$[0, 2], \quad [2, 5], \quad [5, 6].$$

Now insert the points

$$1 \in (0, 2), \qquad 3 \in (2, 5).$$

The resulting set is

$$Q = \{0, 1, 2, 3, 5, 6\}.$$

The partition Q is a **refinement** of P because it contains all the points of P and subdivides each subinterval into smaller ones.

Lower and Upper Sums

Let $f : [a, b] \to \mathbb{R}$ be a **bounded** function and let

$$P = \{x_0, x_1, \dots, x_n\}, \qquad a = x_0 < x_1 < \cdots < x_n = b,$$

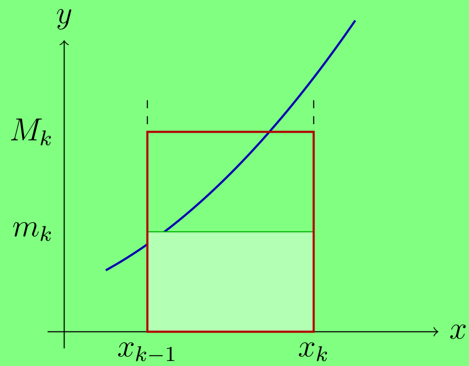
be a partition of $[a, b]$.
For each subinterval $[x_{k-1}, x_k]$, define

$$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}, \qquad M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

The **lower sum** and **upper sum** of f with respect to P are defined by

$$L(f, P) = \sum_{k=1}^{n} m_k \left(x_k - x_{k-1}\right), \qquad U(f, P) = \sum_{k=1}^{n} M_k \left(x_k - x_{k-1}\right).$$

Each term $m_k(x_k - x_{k-1})$ represents the area of a rectangle lying *below* the graph of f, while $M_k(x_k - x_{k-1})$ represents the area of a rectangle lying *above* the graph of f.



Thus, for every partition P, the lower sum underestimates and the upper sum overestimates the area under the curve:

$$L(f, P) \leq U(f, P).$$

Property 1: Lower Sum is Always Below Upper Sum

For every partition P of $[a, b]$,

$$L(f, P) \leq U(f, P).$$

Proof

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. For each subinterval $[x_{k-1}, x_k]$, define

$$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}, \qquad M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

By definition of infimum and supremum,

$$m_k \leq M_k \quad \text{for all } k.$$

Since $x_k - x_{k-1} > 0$, multiplying preserves the inequality:

$$m_k(x_k - x_{k-1}) \leq M_k(x_k - x_{k-1}).$$

Summing over $k = 1, \ldots, n$ gives

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \leq \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = U(f, P).$$

Property 2: Effect of Refinement

If Q is a refinement of a partition P, then

$$L(f, P) \leq L(f, Q), \qquad U(f, Q) \leq U(f, P).$$

Proof

Assume Q is a refinement of P. This means that each subinterval of P is divided into smaller subintervals in Q.
Consider a subinterval $I = [x_{k-1}, x_k]$ of P and let

$$m = \inf\{f(x) \mid x \in I\}.$$

Suppose Q divides I into smaller subintervals $I_1, \ldots, I_r$. For each I_j, let

$$m_j = \inf\{f(x) \mid x \in I_j\}.$$

Since $I_j \subseteq I$, the set of values of f on I_j is smaller, so

$$m \leq m_j \quad \text{for all } j.$$

Multiplying by lengths and summing,

$$m(x_k - x_{k-1}) \leq \sum_{j=1}^{r} m_j |I_j|.$$

Repeating this argument over all subintervals of P yields

$$L(f, P) \leq L(f, Q).$$

A similar argument applies to the upper sums. If

$$M = \sup\{f(x) \mid x \in I\}, \qquad M_j = \sup\{f(x) \mid x \in I_j\},$$

then $M_j \leq M$ for all j, which implies

$$U(f, Q) \leq U(f, P).$$

Property 3: Lower Sum vs Upper Sum for Different Partitions

For any two partitions P_1 and P_2 of $[a, b]$,

$$L(f, P_1) \leq U(f, P_2).$$

Proof

Let

$$R = P_1 \cup P_2.$$

Then R is a refinement of both P_1 and P_2.
By Property 2,

$$L(f, P_1) \leq L(f, R) \quad \text{and} \quad U(f, R) \leq U(f, P_2).$$

By Property 1 applied to R,

$$L(f, R) \leq U(f, R).$$

Combining these inequalities gives

$$L(f, P_1) \leq U(f, P_2).$$

- Lower sums are always below upper sums.
- Refining partitions improves approximation: lower sums increase, upper sums decrease.
- When lower and upper sums approach the same value, that value is the Riemann integral.

Lecture 2: Riemann integrability

Completeness Axiom of $\mathbb{R}$

Every nonempty subset of $\mathbb{R}$ that is bounded above has a **least upper bound** (supremum), and every nonempty subset of $\mathbb{R}$ that is bounded below has a **greatest lower bound** (infimum).

This fundamental axiom is the key ingredient in the definition of the Riemann integral.

Lower and Upper Sums as Sets

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
We associate to f two sets of real numbers:

$$\mathcal{L} := \{ L(f, P) \mid P \text{ is a partition of } [a, b] \},$$
$$\mathcal{U} := \{ U(f, P) \mid P \text{ is a partition of } [a, b] \}.$$

The set $\mathcal{L}$ contains all possible *lower sums* of f, while $\mathcal{U}$ contains all possible *upper sums*.

We now verify that these sets are suitable for taking a supremum and an infimum.

Non-emptiness

Both sets $\mathcal{L}$ and $\mathcal{U}$ are nonempty.

Indeed, at least one partition of $[a, b]$ always exists, for example

$$P_0 = \{a, b\}.$$

Therefore,

$$L(f, P_0) \in \mathcal{L} \qquad \text{and} \qquad U(f, P_0) \in \mathcal{U}.$$

Boundedness

Since f is bounded, there exist real numbers m and M such that

$$m \le f(x) \le M \quad \text{for all } x \in [a, b].$$

For any partition $P = \{x_0, \ldots, x_n\}$, the corresponding lower and upper sums satisfy

$$m(b - a) \le L(f, P) \le U(f, P) \le M(b - a).$$

Consequently:

- the set $\mathcal{L}$ is bounded above,
- the set $\mathcal{U}$ is bounded below.

Definition: Lower and Upper Integrals

By the completeness axiom, the following quantities exist:

Lower integral

$$L(f) := \sup \mathcal{L} = \sup\{\, L(f,P) \mid P \text{ partition of } [a,b] \,\}.$$

Upper integral

$$U(f) := \inf \mathcal{U} = \inf\{\, U(f,P) \mid P \text{ partition of } [a,b] \,\}.$$

Moreover, since $L(f,P) \leq U(f,P)$ for every partition P, we always have

$$L(f) \leq U(f).$$

Riemann Integrability

The function f is called **Riemann integrable** on $[a,b]$ if

$$L(f) = U(f).$$

In this case, the common value is called the **Riemann integral of f over** $[a,b]$ and is denoted by

$$\int_a^b f(x)\,dx.$$

- $L(f)$ is the *best possible lower approximation* of the area under the graph of f.
- $U(f)$ is the *best possible upper approximation*.
- Integrability means that no gap remains between these two quantities.

Lecture 3: The ε-Criterion for Riemann Integrability

The ε–Criterion (Characterization of Integrability)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

The function f is **Riemann integrable** on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

In words: *by choosing the partition fine enough, upper and lower sums can be made arbitrarily close.*

Proof

(⇒) Assume f is integrable.
Let

$$I = \int_a^b f(x)\, dx = L(f) = U(f).$$

Given $\varepsilon > 0$, by the definitions of supremum and infimum, we can find partitions P_1 and P_2 such that

$$L(f, P_1) > I - \frac{\varepsilon}{2}, \qquad U(f, P_2) < I + \frac{\varepsilon}{2}.$$

Let P be a common refinement of P_1 and P_2. Refinement increases lower sums and decreases upper sums, so

$$U(f, P) - L(f, P) \le U(f, P_2) - L(f, P_1) < \varepsilon.$$

(⇐) Assume the ε–condition holds.
For every $\varepsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Since

$$L(f) \ge L(f, P), \qquad U(f) \le U(f, P),$$

we obtain

$$0 \le U(f) - L(f) < \varepsilon.$$

Letting $\varepsilon \to 0$ gives $U(f) = L(f)$, hence f is Riemann integrable.

Example 1: Constant Function

Let $f(x) = c$ on $[a, b]$, where $c \in \mathbb{R}$.

On every subinterval of any partition P,

$$m_k = M_k = c.$$

Thus,

$$L(f, P) = U(f, P) = c(b - a).$$

Therefore,

$$U(f, P) - L(f, P) = 0 < \varepsilon \quad \text{for all } \varepsilon > 0.$$

Hence f is Riemann integrable and

$$\int_a^b f(x)\, dx = c(b - a).$$

Example 2: $f(x) = x^2$

Let $f(x) = x^2$ on $[0, 1]$.

Consider the uniform partition

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\}.$$

Since x^2 is increasing on $[0, 1]$, on each subinterval

$$m_k = \left(\frac{k-1}{n}\right)^2, \qquad M_k = \left(\frac{k}{n}\right)^2.$$

The difference between upper and lower sums satisfies

$$U(f, P_n) - L(f, P_n) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus, for every $\varepsilon > 0$, there exists n such that

$$U(f, P_n) - L(f, P_n) < \varepsilon.$$

By the ε-criterion, f is Riemann integrable and

$$\int_0^1 x^2\, dx = \frac{1}{3}.$$

Theorem: Monotone Functions Are Integrable

Every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof

Assume f is monotone increasing on $[a, b]$. Then f is bounded:

$$f(a) \le f(x) \le f(b) \qquad \text{for all } x \in [a, b].$$

Let $\varepsilon > 0$ be given. Choose

$$\delta > 0 \quad \text{such that} \quad \delta\big(f(b) - f(a)\big) < \varepsilon.$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ satisfying

$$x_k - x_{k-1} < \delta \quad \text{for all } k.$$

Since f is increasing, on each subinterval $[x_{k-1}, x_k]$,

$$m_k = f(x_{k-1}), \qquad M_k = f(x_k).$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} \big(M_k - m_k\big)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big)(x_k - x_{k-1}) \\ &\le \delta \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big) \\ &= \delta\big(f(b) - f(a)\big) \\ &< \varepsilon. \end{aligned}$$

By the ε–criterion, f is Riemann integrable.

Theorem: Continuous Functions Are Integrable

Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof

Since f is continuous on the compact interval $[a, b]$, the **Extreme Value Theorem** implies that f is bounded.
Moreover, by the **Uniform Continuity Theorem**, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition with

$$x_k - x_{k-1} < \delta \quad \text{for all } k.$$

On each subinterval $[x_{k-1}, x_k]$, continuity guarantees the existence of

$$m_k = \min f, \qquad M_k = \max f,$$

and uniform continuity gives

$$M_k - m_k < \frac{\varepsilon}{b - a}.$$

Hence,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &< \frac{\varepsilon}{b - a} \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= \varepsilon. \end{aligned}$$

By the εcriterion, f is Riemann integrable.

Generalization: Finite Discontinuities

Let $f : [a, b] \to \mathbb{R}$ be bounded and have only **finitely many points of discontinuity**. Then f is Riemann integrable.

Interpretation

Continuity everywhere is *sufficient* but not *necessary* for Riemann integrability.

A bounded function can fail to be continuous at finitely many points and still have upper and lower sums that can be made arbitrarily close.

Example (Dirichlet Function).

Define the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function is known as the Dirichlet function and is discontinuous at **every point** in $[0,1]$.
On every subinterval of any partition P of $[0,1]$, we have

$$\inf f = 0, \qquad \sup f = 1.$$

Hence,

$$L(f,P) = 0, \qquad U(f,P) = 1 \quad \text{for all } P.$$

Therefore,

$$U(f,P) - L(f,P) = 1 \not\to 0,$$

and f is not Riemann integrable.

Theorem: Additivity over Subintervals

Let $f : [a,b] \to \mathbb{R}$ be bounded and let $c \in (a,b)$. Then f is Riemann integrable on $[a,b]$ if and only if f is Riemann integrable on both $[a,c]$ and $[c,b]$.
In that case,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof

($\Rightarrow$) Assume f is integrable on $[a,b]$. Restricting any partition of $[a,b]$ to $[a,c]$ and $[c,b]$, and using monotonicity of upper and lower sums, shows that f is integrable on each subinterval.
($\Leftarrow$) Assume f is integrable on both $[a,c]$ and $[c,b]$. Let $\varepsilon > 0$. Choose partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}, \qquad U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$, a partition of $[a,b]$. Then

$$U(f,P) - L(f,P) = \big[U(f,P_1) - L(f,P_1)\big] + \big[U(f,P_2) - L(f,P_2)\big] < \varepsilon.$$

By the ε–criterion, f is integrable on $[a,b]$, and the equality of integrals follows from the definition of sums.

Conventions for Orientation

If f is integrable on $[a, b]$, we define

$$\int_b^a f := -\int_a^b f, \qquad \int_c^c f := 0.$$

With these conventions, for any a, b, c,

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

Theorem: Linearity, Order, and Absolute Value

Let f, g be Riemann integrable on $[a, b]$ and let $k \in \mathbb{R}$.

1. $f + g$ is integrable and
$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$
2. kf is integrable and
$$\int_a^b kf = k\int_a^b f.$$
3. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f \leq \int_a^b g.$$
4. $|f|$ is integrable and
$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof of (1): Linearity

Let $P = \{x_0, \ldots, x_n\}$ be any partition. For each subinterval, define infima and suprema for f, g, and $f + g$.
From properties of infima and suprema,

$$m_k^f + m_k^g \le m_k^{f+g}, \qquad M_k^{f+g} \le M_k^f + M_k^g.$$

Multiplying by Δx_k and summing yields

$$L(f, P) + L(g, P) \le L(f + g, P), \qquad U(f + g, P) \le U(f, P) + U(g, P).$$

Using integrability of f and g, choose a common refinement P so that both upper-lower differences are arbitrarily small. Passing to limits gives

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Proof of (4): Absolute Value Inequality

Since f is integrable, it is bounded. For every partition P,

$$U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P).$$

Hence, $|f|$ is integrable.
Moreover, for all $x \in [a, b]$,

$$-|f(x)| \le f(x) \le |f(x)|.$$

Applying the order property and integrating yields

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|.$$

This implies

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

Lecture 4: Riemann Sum

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. A *partition* of the interval $[a, b]$ is a finite set of points

$$P = \{x_0, x_1, \ldots, x_n\} \quad \text{with} \quad a = x_0 < x_1 < \cdots < x_n = b.$$

For each subinterval $[x_{k-1}, x_k]$, choose a point $\xi_k \in [x_{k-1}, x_k]$. The vector $\xi = (\xi_1, \ldots, \xi_n)$ is called a *tag* (or *mark*) associated with the partition P.

Riemann sum Given a bounded function f, a partition P, and a choice of tags ξ, the *Riemann sum* of f with respect to (P, ξ) is defined by

$$S(f, P, \xi) := \sum_{k=1}^{n} f(\xi_k)\,(x_k - x_{k-1}).$$

Norm of a partition The *norm* (or *mesh*) of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is

$$\|P\| := \max_{1 \le k \le n} (x_k - x_{k-1}).$$

It measures the size of the largest subinterval in the partition.

Theorem (Convergence of tagged Riemann sums). Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition P with $\|P\| \le \delta$ and for every choice of tags ξ, one has

$$\left| S(f, P, \xi) - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Equivalently,

$$\lim_{\|P\| \to 0} S(f, P, \xi) = \int_a^b f(x)\,dx,$$

and the convergence is uniform with respect to the choice of tags ξ.

Proof. For any partition P and any choice of tags ξ, the Riemann sum satisfies the inequalities

$$L(f, P) \le S(f, P, \xi) \le U(f, P),$$

where $L(f, P)$ and $U(f, P)$ denote the lower and upper sums of f corresponding to the partition P.
Since f is Riemann integrable, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|P\| < \delta \quad \Longrightarrow \quad U(f, P) - L(f, P) < \varepsilon.$$

Combining the above inequalities yields

$$\left| S(f, P, \xi) - \int_a^b f(x)\,dx \right| \le U(f, P) - L(f, P) < \varepsilon,$$

which proves the result. □

Exercise. Let $f \in \mathcal{R}[a,b]$. Prove that

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx.$$

Solution. Let $f \in \mathcal{R}[a,b]$. For each $n \in \mathbb{N}$, define the uniform partition

$$P_n: \ a = x_0 < x_1 < \cdots < x_n = b, \qquad x_k := a + k\Delta_n, \qquad \Delta_n := \frac{b-a}{n}.$$

The norm (mesh) of the partition is

$$\|P_n\| = \max_{1\le k\le n} (x_k - x_{k-1}) = \Delta_n = \frac{b-a}{n} \xrightarrow[n\to\infty]{} 0.$$

Choose the right-endpoint tags

$$\xi_k := x_k \in [x_{k-1}, x_k], \qquad k = 1, \dots, n.$$

The corresponding Riemann sum is

$$S(f, P_n, \xi) = \sum_{k=1}^{n} f(\xi_k)\,(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k)\,\Delta_n.$$

Substituting the expressions for x_k and Δ_n yields

$$S(f, P_n, \xi) = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right).$$

Since f is Riemann integrable and $\|P_n\| \to 0$, the convergence theorem for tagged Riemann sums implies

$$\lim_{n\to\infty} S(f, P_n, \xi) = \int_a^b f(x)\,dx.$$

Therefore,

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx.$$

$\square$

Lecture 5: The Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus (FTC)** explains the deep connection between the two central operations of calculus:

$$\text{differentiation} \quad \longleftrightarrow \quad \text{integration.}$$

It shows that, under suitable assumptions, these operations are inverse to each other.

Fundamental Theorem of Calculus

Let $a < b$. The theorem has two complementary parts.

(FTCI: Evaluation of integrals).
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and let $F : [a, b] \to \mathbb{R}$ be differentiable with

$$F'(x) = f(x) \quad \text{for all } x \in [a, b].$$

Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

(FTCII: Differentiation of integrals).
Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable and define

$$G(x) := \int_a^x g(t)\, dt, \qquad x \in [a, b].$$

Then:

- G is continuous on $[a, b]$;
- if g is continuous at $c \in [a, b]$, then G is differentiable at c and

$$G'(c) = g(c).$$

Terminology

- In **FTCI**, the function F is called an **antiderivative** (or primitive) of f.
- In **FTCII**, the function G is called the **indefinite integral** of g.

Proof of FTCI

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem, for each subinterval $[x_{k-1}, x_k]$ there exists $t_k \in (x_{k-1}, x_k)$ such that

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)(x_k - x_{k-1}).$$

Let m_k and M_k be the infimum and supremum of f on $[x_{k-1}, x_k]$. Then

$$m_k \leq f(t_k) \leq M_k,$$

and hence

$$L(f, P) \leq \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \leq U(f, P).$$

Summing the telescoping differences gives

$$\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} \big[F(x_k) - F(x_{k-1})\big] = F(b) - F(a).$$

Since this holds for every partition P, it follows that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof of FTCII

Since g is integrable, it is bounded:

$$|g(x)| \leq M \quad \text{for all } x \in [a, b].$$

Continuity of G. For any $x, y \in [a, b]$,

$$|G(x) - G(y)| = \left| \int_y^x g(t)\,dt \right| \leq \int_y^x |g(t)|\,dt \leq M|x - y|.$$

Thus G is Lipschitz continuous, hence continuous on $[a, b]$.

Differentiability at points of continuity. Assume g is continuous at $c \in [a, b]$. For $x \neq c$,

$$\frac{G(x) - G(c)}{x - c} = \frac{1}{x - c} \int_c^x g(t)\,dt.$$

Given $\varepsilon > 0$, continuity of g at c provides $\delta > 0$ such that $|g(t) - g(c)| < \varepsilon$ whenever $|t - c| < \delta$. Then for $0 < |x - c| < \delta$,

$$\left| \frac{G(x) - G(c)}{x - c} - g(c) \right| = \left| \frac{1}{x - c} \int_c^x (g(t) - g(c))\,dt \right| \leq \varepsilon.$$

Hence $G'(c) = g(c)$.

Remarks

- Not every derivative is continuous, but **every continuous function is the derivative of some function**.
- The FTC allows us to compute definite integrals *without limits or sums*, using antiderivatives.

Consequences of the Fundamental Theorem of Calculus

Mean Value Theorem for Integrals

If $g : [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in (a, b)$ such that

$$\int_a^b g(x)\, dx = (b - a)g(c).$$

Proof

Define $G(x) = \int_a^x g(t)\, dt$. By the FTC, G is differentiable and $G'(x) = g(x)$. Applying the Mean Value Theorem to G on $[a, b]$ yields

$$G'(c) = \frac{G(b) - G(a)}{b - a} = \frac{1}{b - a} \int_a^b g(x)\, dx.$$

Rearranging gives the result.

Substitution Theorem (Change of Variables)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $\varphi : [\alpha, \beta] \to [a, b]$ be continuously differentiable.
Then

$$\int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx.$$

Proof

Let F be an antiderivative of f, so $F' = f$. By the Chain Rule,

$$\frac{d}{dt}\big(F(\varphi(t))\big) = f(\varphi(t))\varphi'(t).$$

Applying the Fundamental Theorem of Calculus gives

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)\, dt = F(\varphi(\beta)) - F(\varphi(\alpha)) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx.$$

Integration by Parts

Let $u, v : [a,b] \to \mathbb{R}$ be differentiable, with continuous derivatives. Then

$$\int_a^b u'(x)v(x)\,dx = \big[u(x)v(x)\big]_a^b - \int_a^b u(x)v'(x)\,dx.$$

Proof

By the Product Rule,

$$(uv)' = u'v + uv'.$$

Integrating both sides on $[a,b]$ and applying the FTC yields

$$\int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a).$$

Rearranging gives the formula.

$$\boxed{\int u'\,v = uv - \int u\,v'}$$

This is the integral version of the Product Rule.

Lecture 6:Improper Integrals

Improper integrals arise when the usual definition of the Riemann integral does not apply directly. This occurs in two main situations:

- one (or both) of the limits of integration is infinite;
- the integrand becomes unbounded at one or more points of the interval.

We study each case separately and then present general criteria for convergence.

Case 1: Integration over an Infinite Interval

Definition. Let $f : [a, \infty) \to \mathbb{R}$ be Riemann integrable on every finite interval $[a, R]$, $R > a$. If the limit

$$\lim_{R\to\infty} \int_a^R f(x)\, dx$$

exists and is finite, then the improper integral is said to *converge*, and we define

$$\int_a^\infty f(x)\, dx := \lim_{R\to\infty} \int_a^R f(x)\, dx.$$

Similarly, if $f : (-\infty, a] \to \mathbb{R}$, we define

$$\int_{-\infty}^a f(x)\, dx := \lim_{R\to-\infty} \int_R^a f(x)\, dx,$$

provided the limit exists.

Example

Consider

$$\int_1^\infty \frac{1}{x^s}\, dx.$$

For $R > 1$,

$$\int_1^R \frac{1}{x^s}\, dx = \begin{cases} \dfrac{1}{s-1}\left(1 - \dfrac{1}{R^{s-1}}\right), & s \neq 1, \\ \log R, & s = 1. \end{cases}$$

Taking the limit as $R \to \infty$, we obtain

$$\int_1^\infty \frac{1}{x^s}\, dx = \begin{cases} \dfrac{1}{s-1}, & s > 1, \\ \text{diverges}, & s \leq 1. \end{cases}$$

Case 2: Unbounded Integrand at an Endpoint

Definition. Let $f : (a, b] \to \mathbb{R}$ be Riemann integrable on $[a+\varepsilon, b]$ for all $\varepsilon > 0$. If the limit

$$\lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)\, dx$$

exists and is finite, then we define

$$\int_a^b f(x)\, dx := \lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^{b} f(x)\, dx.$$

Example

Consider

$$\int_0^1 \frac{1}{x^s}\, dx.$$

For $\varepsilon > 0$ and $s \neq 1$,

$$\int_\varepsilon^1 \frac{1}{x^s}\, dx = \frac{1}{1-s}\big(1 - \varepsilon^{1-s}\big).$$

Letting $\varepsilon \to 0^+$ yields

$$\int_0^1 \frac{1}{x^s}\, dx = \begin{cases} \dfrac{1}{1-s}, & s < 1, \\ \text{diverges}, & s \geq 1. \end{cases}$$

General Definition. Let $f : (a, b) \to \mathbb{R}$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$, be Riemann integrable on every compact subinterval of (a, b). Let $c \in (a, b)$. If both

$$\int_a^c f(x)\, dx \quad \text{and} \quad \int_c^b f(x)\, dx$$

converge, then the improper integral over (a, b) is defined by

$$\int_a^b f(x)\, dx := \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

This definition is independent of the choice of c.

Comparison Test

Let $I = [a, \infty)$ and let $f, g : I \to \mathbb{R}$ be continuous functions such that $0 \leq f(x) \leq g(x)$ for all $x \in I$.

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Definition (Absolute convergence). Let $f : [a, \infty) \to \mathbb{R}$ be integrable on every $[a, R]$. We say that the improper integral $\int_a^\infty f(x)\,dx$ is *absolutely convergent* if

$$\int_a^\infty |f(x)|\,dx \quad \text{converges (as a finite real number).}$$

Theorem (Absolute convergence ⇒ convergence). If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges. Moreover, for every $R > a$,

$$\left| \int_a^R f(x)\,dx \right| \leq \int_a^R |f(x)|\,dx,$$

and in particular (when the limits exist),

$$\left| \int_a^\infty f(x)\,dx \right| \leq \int_a^\infty |f(x)|\,dx.$$

Proof. Define $F(t) = \int_a^t f(x)\,dx$ for $t > a$. To show that $\int_a^\infty f$ converges, it is enough (Cauchy criterion) to show that for every $\varepsilon > 0$ there exists $N > a$ such that for all $b, c > N$,

$$\left| \int_b^c f(x)\,dx \right| < \varepsilon.$$

Assume $\int_a^\infty |f(x)|\,dx$ converges. Then its tails go to 0, i.e., there exists $N > a$ such that for all $b, c > N$,

$$\int_{\min\{b,c\}}^{\max\{b,c\}} |f(x)|\,dx < \varepsilon.$$

Using the triangle inequality for integrals, we get

$$\left| \int_b^c f(x)\,dx \right| \leq \int_b^c |f(x)|\,dx = \int_{\min\{b,c\}}^{\max\{b,c\}} |f(x)|\,dx < \varepsilon.$$

Hence $F(t)$ is Cauchy as $t \to \infty$, so $\lim_{t\to\infty} F(t)$ exists and $\int_a^\infty f(x)\,dx$ converges. The inequalities $\left|\int_a^R f\right| \leq \int_a^R |f|$ (and the limit version) follow from the same estimate. $\square$