Introduction to Real Analysis

Riemann Integral

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Partition

We assume $f : [a, b] \longrightarrow \mathbb{R}$ is bounded.

Definition

A finite order set of points $P = \{x_0, x_1, x_2, \dots, x_n\}$ is called a partition of $[a, b]$ if

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

For $0 \leq i \leq n-1$, let

$$M_i = \sup\{f(x) : x \in [x_i, x_{i+1}]\}$$

$$m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\}$$

Partition

$$M_i = \sup\{f(x) : x \in [x_i, x_{i+1}]\}$$
$$m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\}$$

Upper and Lower sums

Definition

The upper sum $U(f,P)$ and the lower sum $L(f,P)$ of f with respect to P are defined as

$$U(f,P) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i)$$

$$L(f,P) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i)$$

Upper and Lower sums

$$U(f,P)=\sum_{i=0}^{n-1} M_i(x_{i+1}-x_i)$$

$$L(f,P)=\sum_{i=0}^{n-1} m_i(x_{i+1}-x_i)$$

Upper and Lower sums

Note

1. $$L(f,P) \leq U(f,P)$$
2. If $f \geq 0$ then
$$L(f,P) \leq \text{area under } f \leq U(f,P)$$

Partition

Definition

We say a partition Q is finer than P if $P \subset Q$

Examples

1.
$$P = \{x_0, x_1, x_2, \dots, x_n\}$$
$$Q = \{x_0, x_1, u, x_2, \dots, x_n\}$$

2. $P \cup Q$ is finer than both P and Q.

Upper and Lower sums

Lemma

If Q is finer than P then

$$U(f, Q) .... U(f, P)$$

$$L(f, Q) .... L(f, P)$$

Upper and Lower sums

Lemma

If Q is finer than P then

$$U(f,Q)....U(f,P)$$

$$L(f,Q)....L(f,P)$$

$$U(f,Q) \leq U(f,P)$$

$$L(f,Q) \geq L(f,P)$$

Upper and Lower sums

Lemma

For any partitions P and Q of $[a, b]$, we have

$$U(f, P) \ldots . L(f, Q)$$

Upper and Lower sums

Lemma

For any partitions P and Q of $[a, b]$, we have

$$U(f, P) \ldots . L(f, Q)$$

$$U(f, P) \geq L(f, Q)$$

Proof

Upper and Lower sums

Denote the class of all partioins of $[a, b]$ by $\mathcal{P}(a, b)$
If

$$A = \{U(f, P) : P \in \mathcal{P}(a, b)\}$$

$$B = \{L(f, P) : P \in \mathcal{P}(a, b)\}$$

Then A is bounded below by $L(f, P_0)$ for any $P_0 \in \mathcal{P}(a, b)$
therefore, $\inf A$ exists.
Similarly $\sup B$ exists.

Upper and Lower integral

Definition

The upper integral $U(f)$ and the lower integral $L(f)$ of f over $[a,b]$ are defined

$$U(f) = \inf A = \inf\{U(f,P) : P \in \mathcal{P}(a,b)\}$$

$$L(f) = \sup B = \sup\{L(f,P) : P \in \mathcal{P}(a,b)\}$$

Note that

$$L(f) \leq U(f)$$

Riemann integrable

Definition

Let $f : [a, b] \longrightarrow \mathbb{R}$ be bounded. We say that f is Riemann integrable over $[a, b]$ if $L(f) = U(f)$.
In this case, the Riemann integral of f over $[a, b]$ is

$$\int_a^b f = I(f) = U(f) = L(f)$$

The class of functions that are Riemann Integrable over $[a, b]$ are denoted by $\mathcal{R}(a, b)$

$$\int_a^b f = \int_a^b f(x)dx$$

Riemann integrable

Examples

$f : [a, b] \longrightarrow \mathbb{R}$

$$f(x) = c$$

.

Dirichlet fucntion

Examples

$f : [a, b] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b] \\ 0 & x \in \mathbb{Q}^c \cap [a, b] \end{cases}$$

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Riemann's Criterion

Theorem

The following statements are equivalent

1. $f \in \mathcal{R}(a,b)$
2. For all $\varepsilon > 0$, there is a partition $P \in \mathcal{P}(a,b)$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

Riemann integrable

Corollary

$f \in \mathcal{R}(a,b)$ iff there exists a sequence (P_n) in $\mathcal{P}(a,b)$ such that

$$U(f,P_n) - L(f,P_n) \longrightarrow 0$$

In this case,

$$\int_a^b f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$$

Riemann integrable

Examples

$f : [a, b] \longrightarrow \mathbb{R}$

$$f(x) = x$$

.

Norm of Partition

Definition

The norm of the partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is

$$||P|| = \max\{x_{i+1} - x_i : i = 0, 1, \ldots, n-1\}$$

That is $||P||$ is the length of the longest subinterval of P.

Riemann integrable

Theorem

1. If f is monotonic on $[a, b]$, then $f \in \mathcal{R}(a, b)$
2. If f is continous on $[a, b]$, then $f \in \mathcal{R}(a, b)$

Darboux's Theorem

Darboux's Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ be bounded. Given $\varepsilon > 0$, there is a $\delta > 0$ such that
If P is any partition of $[a, b]$ satisfies $||P|| < \delta$ then

$$U(f, P) - U(f) < \varepsilon$$

and

$$L(f) - L(f, P) < \varepsilon$$

i.e.,

$$\lim_{||P|| \to 0} U(f, P) = U(f)$$

$$\lim_{||P|| \to 0} L(f, P) = L(f)$$

Riemann Sum

Definition

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. We say that $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ is a mark on P if $\alpha_i \in [x_i, x_{i+1}]$ for all $i \in \{0, 1, \ldots, n-1\}$, then the sum

$$S(f, P, \alpha) = \sum_{i=0}^{n-1} f(\alpha_i)(x_{i+1} - x_i)$$

is the Riemann sum of f on P with mark α.

$$L(f, P) \leq S(f, P, \alpha) \leq U(f, P)$$

Riemann Sum

Theorem

The following statements are equivalent

1. $f \in \mathcal{R}(a,b)$ with integral equals A
2. For any $\varepsilon > 0$, there is a $\delta > 0$ such that, it P is any partition satisfies $||P|| < \delta$ and α is any mark on P then

$$|S(f,P,\alpha) - A| < \varepsilon$$

i.e.,

$$\lim_{||P|| \to 0} S(f,P,\alpha) = A$$

Riemann Sum

Corollary

If $f \in \mathcal{R}(a,b)$ and (P_n) is a sequence of partitions such that $||P_n|| \to 0$, then for any choice of marks α_n on P_n, we have

$$\int_a^b f = \lim_{n\to\infty} S(f, P_n, \alpha)$$

Riemann Sum

Examples

Evaluate

$$\int_0^1 (x - x^2)dx$$

.

Properties of Integral

Theorem

If $f, g \in \mathcal{R}(a, b)$ and $c, d \in \mathbb{R}$ then

1. $$\int_a^b cf + dg = c\int_a^b f + d\int_a^b g$$
2. If $f(x) \geq 0$ for all $x \in [a, b]$ then
$$\int_a^b f \geq 0$$
3. If $f(x) \leq g(x)$ for all $x \in [a, b]$ then then
$$\int_a^b f \leq \int_a^b g$$

Properties of Integral

Theorem

If $f \in \mathcal{R}(a,b)$ and $c \in (a,b)$ then

1. $$\int_a^b f = \int_a^c f + \int_c^b f$$
2. If $f(x) \geq 0$ for all $x \in [a,b]$ and f is continuous, then $\int_a^b f = 0$ iff $f(x) = 0$ for all $x \in [a,b]$
3. $$\int_a^b f = -\int_b^a f$$
4. $$\int_a^a f = 0$$

The Mean Value Theorem

Theorem

If f is continuous on $[a, b]$ then there exists $c \in (a, b)$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

.

Properties of Integral

Theorem

1. If $f : [a, b] \to [c, d]$ is Riemann integrable, and $g : [c, d] \to \mathbb{R}$ is continuous then $g \circ f$ is Riemann integrable on $[a, b]$.
2. If $f \in \mathcal{R}(a, b)$ then $|f| \in \mathcal{R}(a, b)$ and
$$\left| \int_a^b f \right| = \int_a^b |f|$$
3. If $f \in \mathcal{R}(a, b)$ then $f^n \in \mathcal{R}(a, b)$ for all $n \in \mathbb{N}$
4. If $f, g \in \mathcal{R}(a, b)$ then $fg \in \mathcal{R}(a, b)$.

The Fundamental Theorem of Calculus

Theorem

If $f \in \mathcal{R}(a,b)$ and let $F : [a,b] \to \mathbb{R}$ be defined by

$$F(x) = \int_a^x f(t)dt$$

then

1. F is continuous on $[a,b]$
2. If f is continuous, then F is differentiable and

$$F' = f$$

The Fundamental Theorem of Calculus

Theorem

If F is differentiable on $[a, b]$ and $F' \in \mathcal{R}(a, b)$ then

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Substitution Rule

Theorem

Suppose φ is differentiable on $[a, b]$ and φ' is continuous. If f is continuous on the range of φ, then

$$\int_a^b f(\varphi(t))\varphi'(t)dt = \int_{\varphi(a)}^{\varphi(b)} f(x)dx$$

Substitution Rule

Examples

$$\int_1^2 t\sqrt{t^2+1}\,dt$$

.

Substitution Rule

Theorem

Suppose φ is differentiable on $[a, b]$ and φ' is continuous and $\varphi(x) \neq 0$ for all $x \in (a, b)$.
If f is continuous on the range of φ, and ψ is the inverse of φ then

$$\int_a^b f(\varphi(t))dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\psi'(x)dx$$

Substitution Rule

Examples

$$\int_1^4 \frac{1}{(1+\sqrt{t})^2}dt$$

.

Integration by parts

Theorem

Let $f, g; [a, b] \to \mathbb{R}$ be differentiable. If $f', g' \in \mathcal{R}(a, b)$ then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Improper Integral

Definition

If f is defined on $[a, b]$ and $\lim_{x\to a^+} f(x) = \infty$ or $-\infty$ and $f \in \mathcal{R}(c, b)$ for all $c \in (a, b)$, then the improper integral

$$\int_{a^+}^{b} f$$

is defined by

$$\lim_{c\to a^+} \int_{c}^{b} f$$

If the limit exists, then it represents the value of the proper integral.

$$\int_{a^+}^{b} f = \int_{a}^{b} f$$

Improper Integral

Examples

$f : (0, 1] \to \mathbb{R},\ f(x) = \frac{1}{\sqrt{x}}$

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Improper Integral

Examples

$f : (0,1] \to \mathbb{R},\ f(x) = \frac{1}{x^3}$

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Improper Integral

Definition

If $\lim_{x\to b^-} f(x) = \infty$ or $-\infty$ and $f \in \mathcal{R}(a,c)$ for all $c \in (a,b)$ then

$$\int_a^{b^-} f = \lim_{c\to b^-} \int_a^c f$$

If the limit exists

Improper Integral

Definition

If $c \in (a, b)$ and f is unbounded in the neighborhood of c then

$$\int_a^b f = \int_a^{c^-} f + \int_{c^+}^b f$$

If both integrals exist.

Improper Integral

Definition

If $f : [a, \infty) \to \mathbb{R}$ is Riemann integrable on $[a, c]$ for all $c > a$ and

$$\lim_{c \to \infty} \int_a^c f$$

exists then the improper integral

$$\int_a^\infty f = \lim_{c \to \infty} \int_a^c f$$

Improper Integral

Definition

Similarly

$$\int_{-\infty}^{a} f = \lim_{c \to -\infty} \int_{c}^{a} f$$

if the limit exists

Definition

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{a} f + \int_{a}^{\infty} f$$

if both integrals exist if the limit exists

Improper Integral

Examples

$$\int_1^\infty \frac{1}{x^2} dx$$

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Improper Integral

Examples

$$\int_{-\infty}^{\infty} x dx$$

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