Revision For Math. 111
2nd Mid-Term Exam

Q1 Evaluate: $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx$

Q2 Evaluate: $\int \frac{1}{2+\cos x}\,dx$

Q3 Evaluate: $\int \frac{x^{10}}{x^4-1}\,dx$

Q4 Evaluate: $\int \tan^{-1} x\,dx$

Q5 Find y' for each of the following:

i) $y=5\sinh^{-1}(\sqrt{x})+\operatorname{sech}^{-1}x$

ii) $y=\cosh\left(\frac{1}{x}\right)+\tanh^{-1}(x^2)$

Q6 Evaluate: $\int \tan^5 x\,dx$

Q7 Evaluate: $\int \sin^4 x\,dx$

Q8 Evaluate each of the following:

i) $\int \sqrt{25-x^2}\,dx$

ii) $\int x\sqrt{x^2+4}\,dx$

Q9 Evaluate: $\int \tan^3 x\sec^5 x\,dx$

Q10 Evaluate: $\int \sin^5 x\cos^2 x\,dx$

Q11 Evaluate: $\int \frac{1}{x\sqrt{4-x^2}}\,dx$

1

Model Answer

Q1 $I = \int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx$

let $t^3 = 1 + \sqrt[4]{x}$

$\Rightarrow \sqrt[4]{x} = t^3 - 1$

$\Rightarrow x = (t^3-1)^4$

$\Rightarrow \sqrt{x} = (t^3-1)^2, \quad dx = 12(t^3-1)^3 t^2\,dt$

$I = 12\int \frac{t^3 \cancel{(t^3-1)^3}\,dt}{\cancel{(t^3-1)^2}}$

$I = 12\int (t^6 - t^3)\,dt$

$I = 12\left[\frac{t^7}{7} - \frac{t^4}{4}\right] + C$

$I = 12\left[\frac{(1+\sqrt[4]{x})^{7/3}}{7} - \frac{(1+\sqrt[4]{x})^{4/3}}{4}\right] + C$

Q2 $I = \int \frac{1}{2+\cos x}\,dx$

let $u = \tan(x/2)$

$\Rightarrow dx = \frac{2}{1+u^2}\,du, \quad \cos x = \frac{1-u^2}{1+u^2}$

$I = \int \frac{\frac{2}{\cancel{1+u^2}}}{\frac{2+2u^2+1-u^2}{\cancel{1+u^2}}}$

$I = \int \frac{2}{3+u^2}\,du$

$I = 2\int \frac{du}{3+u^2}$

$I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$

$I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$

Q3 $I = \int \frac{x^{10}}{x^4-1}\,dx$

By long Division & factorization

$\Rightarrow \frac{x^{10}}{x^4-1} = x^6 + x^2 + \frac{x^2}{(x-1)(x+1)(x^2+1)}$

$\frac{x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$\Rightarrow A = 1/4, \quad B = -1/4, \quad C = 0, \quad D = 1/2$

$I = \int \left[x^6 + x^2 + \frac{1/4}{x-1} - \frac{1/4}{x+1} + \frac{1/2}{x^2+1}\right] dx$

$\Rightarrow I = \frac{x^7}{7} + \frac{x^3}{3} + \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\tan^{-1}x + C$

Q4 $\int \tan^{-1}x\,dx$

$u = \tan^{-1}x \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx, \qquad v = x$

$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x\,dx}{1+x^2}$

$\therefore \int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$

Q5 i) $y = 5\sinh^{-1}\sqrt{x} + \operatorname{sech}^{-1}x$

$$y' = \frac{5}{\sqrt{1+x}} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{x\sqrt{1-x^2}}$$

$$y' = \frac{5}{2\sqrt{x}\sqrt{1+x}} - \frac{1}{x\sqrt{1-x^2}}$$

ii) $y = \cosh\left(\frac{1}{x}\right) + \tanh^{-1}(x^2)$

$$y' = \sinh\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) + \frac{1}{1-x^4} \cdot 2x$$

$$y' = \frac{-1}{x^2}\sinh\left(\frac{1}{x}\right) + \frac{2x}{1-x^4}$$

Q6 $I = \int \tan^5 x\,dx$

$$I = \int \tan^3 x \tan^2 x\,dx$$

$$I = \int \tan^3 x(\sec^2 x - 1)\,dx$$

$$I = \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx$$

$$I = \int \tan^3 x \sec^2 x\,dx - \int \tan x(\sec^2 x - 1)\,dx$$

$$I = \int \tan^3 x \sec^2 x\,dx - \int \tan x \sec^2 x\,dx + \int \tan x\,dx$$

$$I = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C$$

Q7 $I = \int \sin^4 x\,dx$

$$I = \frac{1}{4}\int (1 - \cos 2x)^2\,dx$$

$$I = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$I = \frac{1}{4}\int \left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]dx$$

$$I = \frac{1}{4}\left(x - \sin 2x + \frac{1}{2}x + \frac{\sin 4x}{8}\right) + C$$

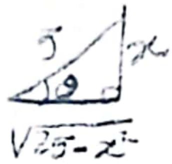
$$I = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Q8 (i) $\int \sqrt{25 - x^2}\,dx$

let $x = 5\sin\theta \Rightarrow dx = 5\cos\theta\,d\theta$

$$I = \int \sqrt{25 - 25\sin^2\theta} \cdot 5\cos\theta\,d\theta$$

$$I = 25\int \cos^2\theta\,d\theta$$



$$I = \frac{25}{2}\int (1 + \cos 2\theta)\,d\theta$$

$$I = \frac{25}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

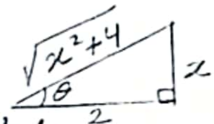
$$I = \frac{25}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$I = \frac{25}{2}\left[\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25 - x^2}}{25}\right] + C$$

ii) $\int x\sqrt{x^2 + 4}\,dx$

let $x = 2\tan\theta$

$\Rightarrow dx = 2\sec^2\theta\,d\theta$



$$I = \int 2\tan\theta \cdot 2\sec\theta \cdot 2\sec^2\theta\,d\theta$$

$$I = 8\int \sec^2\theta \cdot \sec\theta\tan\theta\,d\theta$$

$$I = 8\left(\frac{\sec^3\theta}{3}\right) + C$$

$$I = \frac{\cancel{8}}{3} \cdot \frac{(\sqrt{x^2 + 4})^3}{\cancel{8}} + C$$

$$I = \frac{1}{3}(x^2 + 4)^{3/2} + C$$

Q9 $I = \int \tan^3 x \sec^5 x \, dx$

let $\tan^2 x = \sec^2 x - 1$

$, u = \sec x \Rightarrow du = \sec x \cdot \tan x \, dx$

$I = \int (\sec^2 x - 1) \sec^4 x \cdot \sec x \tan x \, dx$

$I = \int (u^2 - 1) u^4 \, du$

$I = \int (u^6 - u^4) \, du$

$I = \frac{u^7}{7} - \frac{u^5}{5} + C$

$I = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$

Q10 $I = \int \sin^5 x \cos^2 x \, dx$

$I = -\int \sin^4 x \cos^2 x \cdot (-\sin x) \, dx$

$I = -\int (1 - \cos^2 x)^2 \cos^2 x \cdot (-\sin x) \, dx$

$\sin^2 x = 1 - \cos^2 x$
$, u = \cos x, du = -\sin x \, dx$

$I = -\int (1 - u^2)^2 u^2 \, du$

$I = -\int (1 - 2u^2 + u^4) u^2 \, du$

$I = -\int (u^2 - 2u^4 + u^6) \, du$

$I = -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C$

$I = -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$

Q11 $\int \frac{1}{x\sqrt{4 - x^2}} \, dx$

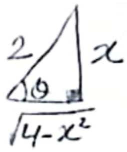
$I = \int \frac{1}{x\sqrt{4 - x^2}} \, dx$

$I = \frac{-1}{2} \operatorname{sech}^{-1}\left(\frac{|x|}{2}\right) + C$

$, \quad 0 < |x| < 2$

OK

Another soln



let $x = 2\sin\theta$

$\Rightarrow dx = 2\cos\theta \, d\theta$

$I = \int \frac{\cancel{2\cos\theta} \, d\theta}{2\sin\theta \cdot \cancel{2\cos\theta}}$

$I = \frac{1}{2} \int \csc\theta \, d\theta$

$I = \frac{1}{2} \ln |\csc\theta - \cot\theta| + C$

$I = \frac{1}{2} \ln \left[\frac{2}{x} - \frac{\sqrt{4 - x^2}}{x} \right] + C$

$, \quad 0 < x < 2$

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