Functional Analysis Exercise Class

Week November 30 – Dec 4:

Deadline to hand in the homework: your exercise class on week December 7 – 11

Exercises with solutions

Recall that every normed space X can be isometrically embedded into its bidual by the map $(Jx)(\varphi) := \varphi(x)$, $x \in X$, $\varphi \in X^*$, and X is called reflexive if J is a bijection.

(1) Show that $l_p := l_p(\mathbb{N}, \mathbb{K})$ is reflexive for every $p \in (1, +\infty)$.

Solution: Note that J is injective, and hence we only have to show surjectivity.

Recall (from the previous classes) that for every $\varphi \in l_p^*$, there exists a unique $y_\varphi \in l_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that $\|y_\varphi\|_q = \|\varphi\|$, and

$$\sum_{n \in \mathbb{N}} y_\varphi(n) x(n) = \varphi(x) = (Jx)(\varphi), \qquad x \in l_p.$$

Now let $f \in l_p^{**}$. Since $l_p^* \cong l_q$, we have $l_p^{**} \cong l_q^*$, and we can consider f to be a bounded linear functional on l_q. Hence, by the above, there exists a unique $z_f \in l_p$ such that

$$f(y_\varphi) = \sum_{n \in \mathbb{N}} z_f(n) y_\varphi(n) = \varphi(z_f) = (Jz_f)(y_\varphi).$$

Since every $y \in l_q$ is equal to y_φ for some $\varphi \in l_p^*$, this shows that $Jz_f = f$. That is, every $f \in l_p^{**}$ can be obtained in the form $f = Jx$ for some $x \in l_p$, and hence J is surjective.

(2) Let X be a normed space. Prove that

a) If X is finite-dimensional then it is reflexive.

b) If X is reflexive and separable then X^* is separable.

c) If X is reflexive then X^* is reflexive.

d) If X is a Banach space and X^* is reflexive then X is reflexive.

Solution:

a) We know that the canonical embedding $J : X \to X^{**}$ is injective. If X is finite-dimensional then $\dim X = \dim X^* = \dim X^{**}$, and hence an injective map from X to X^{**} is also surjective. Therefore J is also surjective, proving that X is reflexive.

b) If X is reflexive then X^{**} is isometrically isomorphic to X under the canonical embedding J. Hence, if $\mathcal{D}$ is a countable dense set in X then $J(\mathcal{D})$ is a countable dense set in X^{**}, and therefore if X is separable then so is X^{**}. We have seen in the lecture that if the dual space of a normed space is separable then so is the space itself. Since X^{**} is the dual of X^*, the assertion follows from the above observations.

c) Let $J_X : X \to X^{**}$ and $J_{X^*} : X^* \to (X^*)^{**} = X^{***}$ be the canoncial embeddings of X and X^* into their biduals, respectively. We have to show that if $F \in X^{***}$ then there exists an $f_F \in X^*$ such that $J_{X^*} f_F = F$.

Since X is reflexive, for every $\varphi \in X^{**}$ there exists a unique $x_\varphi \in X$ such that $\varphi = J_X x_\varphi$. For the above F and $\varphi \in X^{**}$, we have

$$F(\varphi) = F(J_X x_\varphi) = (F \circ J_X)(x_f). \tag{0.1}$$

Note that $F \circ J_X \in X^*$, and we have, for any $\varphi \in X^{**}$,

$$J_{X^*}(F \circ J_X)\varphi = \varphi(F \circ J_X) = (J_X x_\varphi)(F \circ J_X) = (F \circ J_X)(x_\varphi) = F(\varphi),$$

where in the last step we used (0.1). Hence, $F = J_{X^*}(F \circ J_x)$, as required.

d) Assume, on the contrary, that X^* is reflexive but X is not. That means that there exists a $\varphi \in X^{**} \setminus J_X(X)$. Since X is a Banach space, and J_X is an isometric isometry, $J_X(X)$ is also a Banach space, and therefore it is closed. Hence, by the spanning criterion (see the lecture), there exists an $F \in X^{***}$ such that $F|_{J_X(X)} = 0$ and $F(\varphi) \neq 0$. Since X^* is reflexive, there exists a (unique) $f \in X^*$ such that $F = J_{X^*} f$. Hence, for every $x \in X$,

$$0 = F(J_X x) = (J_{X^*} f)(J_X x) = (J_X x)(f) = f(x),$$

and therefore $f = 0$. However, this implies $F = 0$, which contradicts $F(\varphi) \neq o$.

(3) Let X, Y be normed spaces and $T : X \to Y$ a linear operator. Show that the following are equivalent.

(i) The graph $\Gamma(T)$ of T is closed.

(ii) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \to 0$ and $(Tx_n)_{n\in\mathbb{N}}$ converges, then $\lim_{n\to+\infty} Tx_n = 0$.

Solution: (i)$\Rightarrow$(ii): We have $\Gamma(T) \ni (x_n, Tx_n) \to (0, y)$. Since $\Gamma(T)$ is closed, we have $(0, y) \in \Gamma(T)$, i.e., $y = T(0)$. Since T is linear, we have $y = T(0) = 0$.

(ii)$\Rightarrow$(i): $\Gamma(T)$ is closed if and only if every convergent sequence in $\Gamma(T)$ has its limit in $\Gamma(T)$. Let $\Gamma(T) \ni (x_n, Tx_n) \to (x, y)$ as $n \to +\infty$. We have to show that $(x, y) \in \Gamma(T)$.

We have $\Gamma(T) \ni (x_n - x, T(x_n - x)) \to (0, y - T(x))$. By assumption (ii), $y - T(x) = 0$, and hence $(x, y) \in \Gamma(T)$, as required.

(4) Let X, Y be Banach spaces, and $T_n \in \mathcal{B}(X, Y)$, $n \in \mathbb{N}$, be a sequence of bounded operators that is pointwise convergent, i.e., for every $x \in X$, $(T_n(x))_{n\in\mathbb{N}}$ is convergent. Show that $Tx := \lim_{n\to+\infty} T_n x$, $x \in X$, defines a bounded linear operator.

(Hint: Use the uniform boundedness theorem.)

Solution: It is clear from the definition that T is linear. By assumption, for every $x \in X$, $\lim_{n\to+\infty} \|T_n x\| = \|Tx\|$; in particular, $\sup_{n\in\mathbb{N}} \|T_n x\| < +\infty$. Hence, by the uniform boundedness theorem, $M := \sup_{n\in\mathbb{N}} \|T_n\| < +\infty$. Thus,

$$\|Tx\| = \lim_{n\to+\infty} \|T_n x\| \leq \|x\| \sup_{n\in\mathbb{N}} \|T_n\| = M \|x\|, \qquad x \in X,$$

and therefore T is bounded with $\|T\| \leq \sup_{n\in\mathbb{N}} \|T_n\|$.

(5) By definition, the weak topology $\sigma(X, X^*)$ on a normed space X is the weakest topology on X such that all elements of X^* are continuous w.r.t it. Show that the weak topology is generated by the sets

$$U_{f,c,\varepsilon} := f^{-1}(D_\varepsilon(c)) = \{x \in X : |f(x) - c| < \varepsilon\}, \qquad c \in \mathbb{K}, \varepsilon > 0, f \in X^*,$$

where $D_\varepsilon(c) := \{d \in \mathbb{K} : |d - c| < \varepsilon\}$ is the open ball of radius $\varepsilon > 0$ around c in $\mathbb{K}$.

Solution: Since $D_\varepsilon(c)$ is open and f is continuous w.r.t. the weak topology, $U_{f,c,\varepsilon} \in \sigma(X, X^*)$ for every $c \in \mathbb{K}$, $\varepsilon > 0$, $f \in X^*$. Hence, the topology τ generated by these sets satisfies $\tau \subseteq \sigma(X, X^*)$. On the other hand, every open set $U \subseteq \mathbb{K}$ can be written as $U = \cup_{c \in U} D_{\varepsilon_c}(c)$ with some $\varepsilon_c > 0$, $c \in U$, and hence

$$f^{-1}(U) = \bigcup_{c \in U} f^{-1}(D_{\varepsilon_c}(c)) = \bigcup_{c \in U} U_{f,c,\varepsilon}.$$

Thus $f^{-1}(U) \in \tau$ for any open set $U \subseteq \mathbb{K}$ and thus f is continuous w.r.t. τ. Hence $\tau \supseteq \sigma(X, X^*)$.

(6) Let X be a normed space, and let $\mathcal{P}_f(X^*)$ denote the finite subsets of X^*. For every $\mathcal{F} \in \mathcal{P}_f(X^*)$, let

$$|x|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |f(x)|.$$

a) Show that for every $\mathcal{F} \in \mathcal{P}_f(X^*)$, $|\ |_{\mathcal{F}}$ is a seminorm, i.e., for every $x, y \in X$, $\lambda \in \mathbb{C}$, (i) $|x|_{\mathcal{F}} \geq 0$; (ii) $|\lambda x|_{\mathcal{F}} = |\lambda||x|_{\mathcal{F}}$; (iii) $|x + y|_{\mathcal{F}} \leq |x|_{\mathcal{F}} + |y|_{\mathcal{F}}$.

b) Show that $U \subset X$ is open in the weak topology $\sigma(X, X^*)$ if and only if

(P) for every $x \in U$ there exists an $\mathcal{F} \in \mathcal{P}_f(X^*)$ and $\varepsilon > 0$ such that

$$B_{\mathcal{F}}(x, \varepsilon) := \{y \in X : |y - x|_{\mathcal{F}} < \varepsilon\} \subseteq U.$$

c) Show that the weak topology makes X a topological vector space, i.e., the addition $+ : X \times X \to X$ and the scalar multiplication $\cdot : \mathbb{K} \times X \to X$ are continuous, where on product spaces we use the product topology.

Solution:

a) Properties (i) and (ii) are trivial from the definition. For $x, y \in X$, we have

$$\begin{aligned} |x + y|_{\mathcal{F}} &= \sup_{f \in \mathcal{F}} |f(x + y)| \leq \sup_{f \in \mathcal{F}} \{|f(x)| + |f(y)|\} \leq \sup_{f,g \in \mathcal{F}} \{|f(x)| + |g(y)|\} \\ &= \sup_{f \in \mathcal{F}} |f(x)| + \sup_{g \in \mathcal{F}} |g(y)| = |x|_{\mathcal{F}} + |y|_{\mathcal{F}}. \end{aligned}$$

b) We have

$$B_{\mathcal{F}}(x, \varepsilon) := \bigcap_{f \in \mathcal{F}} f^{-1}(\{c \in \mathbb{K} : |f(x) - c| < \varepsilon\}) = \bigcap_{f \in \mathcal{F}} f^{-1}(D_\varepsilon(f(x))).$$

Since every $f \in \mathcal{F}$ is continuous w.r.t. weak topology, $B_{\mathcal{F}}(x, \varepsilon)$ is the finite intersection of w-open sets, and therefore is itself w-open. Thus, if U is a set with property (P) then every point of U is a w-interior point, and hence U is w-open.

Let τ be the collection of all sets with the (P) property. It is easy to see that τ is a topology, and by the above, $\tau \subseteq \sigma(X, X^*)$. On the other hand, τ contains $f^{-1}(\{d : |d - c| < \varepsilon\})$ for every $c \in \mathbb{K}$ and $\varepsilon > 0$. Since $\sigma(X, X^*)$ is a topology generated by these sets, according to Exercise (5), we get $\tau \supseteq \sigma(X, X^*)$. Therefore, U is w-open if and only if U has property (P).

c) Let $x, y \in X$. By the previous point, continuity of the addition at (x, y) is equivalent to the following: For every $\mathcal{F} \in \mathcal{P}_f(X^*)$ and every $\varepsilon > 0$ there exist $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{P}_f(X^*)$ and $\delta_1, \delta_2 > 0$ such that for all $x' \in B_{\mathcal{F}_1}(x, \delta_1)$, $y' \in B_{\mathcal{F}_2}(y, \delta_2)$, we have $x' + y' \in B_{\mathcal{F}}(x + y, \varepsilon)$. Choosing $\mathcal{F}_1 := \mathcal{F}_2 := \mathcal{F}$ and $\delta_1 := \delta_2 := \varepsilon/2$, we get

$$|(x' + y') - (x + y)|_{\mathcal{F}} \le |x' - x|_{\mathcal{F}} + |y' - y|_{\mathcal{F}} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Continuity of the scalar multiplication follows by a similar argument.

(7) Show that on any finite-dimensional normed space the weak topology coincides with the topology generated by any norm.

Solution: Let X be a finite-dimensional vector space, let $e_1, \ldots, e_d$ be a basis in X, and let $f_1, \ldots, f_d$ be its dual basis, defined by $f_i(e_j) := \delta_{i,j}$. Then $\|x\|_\infty := \max_{1 \le i \le d} |f_i(x)|$ is a norm on X, and, since X is finite-dimensional, all linear functionals on X are also continuous.

We know that on a finite-dimensional vector space any two norms are equivalent, so it is enough to compare the weak topology to the topology τ induced by $\| \ \|_\infty$. It is clear that $\tau \supseteq \sigma(X, X^*)$. On the other hand,

$$|x|_{\{f_1, \ldots, f_d\}} = \sup_{1 \le i \le d} |f_i(x)| = \|x\|_\infty, \qquad x \in X,$$

and hence the open $\| \ \|_\infty$-balls around any point and with any radius are open in the weak topology, according to Exercise (6). Hence, $\tau \subseteq \sigma(X, X^*)$.

(8) Let X be an inifinite-dimensional normed space and $S_X := \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Show the following (maybe) surprising fact: The closure of the unit sphere in the weak topology is the whole closed unit ball, i.e.,

$$\overline{\{x \in X : \|x\| = 1\}}^{\sigma(X, X^*)} = \{x \in X : \|x\| \le 1\}.$$

Conclude that the weak topology and the norm topology are different on any inifinite-dimensional normed space.

(Hint: Use the following fact, proved in the lecture: if x is in the weak interior of a set U then there is a non-zero vector $z \in X$ such that $x + cz \in U$ for every $c \in \mathbb{K}$.)

Solution: Let $\|x\| > 1$. By the Hahn-Banach theorem, there exists an $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Hence, $\{y \in X : |f(y)| > (1 + \|x\|)/2\}$ is a w-open set that contains x but is disjoint from the unit sphere (since $|f(z)| \le \|z\| \le 1$, $z \in S_X$), and hence x is not in the weak closure of S_X.

On the other hand, let $\|x\| < 1$, and let $U \in \sigma(X, X^*)$ be a w-open set that contains x. By the statement in the Hint, there exists a non-zero vector $z \in X$ such that $x + cz \in U$ for every $c \in \mathbb{K}$. Let $g(t) := \|x + tz\|$, $t \in \mathbb{R}$. Then g is continuous, $g(0) < 1$, and by the triangle inequality, $g(t) \geq |t| \|z\| - \|x\|$, thus $\lim_{t \to +\infty} g(t) = +\infty$. Hence, by the Bolzano-Weierstrass theorem, there exists a $t \in \mathbb{R}$ such that $x + tz \in U$ and $\|x + tz\| = 1$, i.e., $x + tz \in U \cap S_X$. This shows that every open neighbourhood of x intersects S_X, and thus x is in the closure of S_X.

Since S_X is closed in the norm topology, but not in the weak topology, as we have seen above, the two topologies have to be different.

(9) *a)* Show that every weakly convergent sequence in $l_1 := l_1(\mathbb{N}, \mathbb{K})$ is norm convergent.

b) Decide whether the following statement is true or false: A set M is closed in the weak topology of l_1 if and only if every convergent sequence in M has its limit point in M.

Solution:

a) Suppose for contradiction that $(x^n)_{n\in\mathbb{N}} \in (l^1)^{\mathbb{N}}$ is weakly convergent but not norm convergent. We can assume without loss of generality that $(w) \lim_{n\to+\infty} x^n = 0$ (otherwise consider the sequence $(x^n - x)$ instead). The fact that the sequence is not convergence is equivalent to a positive $\delta > 0$ and the existence of a subsequence $x^{k(n)}$ such that $\left\|x^{k(n)}\right\|_1 \geq \delta$, $n \in \mathbb{N}$. By restricting to this subsequence, we can assume that $\left\|x^{k(n)}\right\|_1 \geq \delta$ holds for every $n \in \mathbb{N}$.
By the definition of the weak convergence, we have $f(x^n) \to 0$ for any $f \in (l_1)^*$. Thus, also for each coordinate function $f_k : l_1 \to \mathbb{C}$ defined as $f_k(x) = x_k$ (which is clearly bounded) we have $f_k(x^n) \to 0$ as $n \to \infty$, i.e. $x_k^n \to 0$ for every $k \in \mathbb{N}$. As $x^1 \in l_1$ we can choose $K_1 \in \mathbb{N}$ with $\sum_{k=K_1+1}^{\infty} |x_k^1| < \frac{\delta}{5}$. As $\sum_{k=1}^{K_1} x_k^n \to 0$ for $n \to \infty$ there exists an $n_2 \in \mathbb{N}$ with $\sum_{k=1}^{K_1} |x_k^{n_2}| < \frac{\delta}{5}$. Now we may again choose $K_2 \in \mathbb{N}$ such that $\sum_{k=K_2+1}^{\infty} |x_k^{n_2}| < \frac{\delta}{5}$ and continue the construction from above. Repeating this argument leads to a subsequence $(x^{n_j})_{j\in\mathbb{N}}$ and a sequence $(K_j)_{j\in\mathbb{N}}$ of integers such that $\sum_{k=1}^{K_{j-1}} |x_k^{n_j}| < \frac{\delta}{5}$ and $\sum_{k=K_j+1}^{\infty} |x_k^{n_j}| < \frac{\delta}{5}$.
Now define

$$y_k = \begin{cases} 1 & \text{if } x_k^{n_j} = 0 \\ \frac{|x_k^{n_j}|}{x_k^{n_j}} & \text{if } x_k^{n_j} \neq 0 \end{cases}.$$

for $K_{j-1} < k \leq K_j$.
We have $y \in l^\infty$ as $|y_k| = 1$ for all $k \in \mathbb{N}$ and therefore we can define an $f \in X^*$ via

$f(x) = \sum_{k=1}^{\infty} y_k x_k$. For this functional we have:

$$\begin{aligned}
|f(x^{n_j})| = \left|\sum_{k=1}^{\infty} y_k x_k^{n_j}\right| &\geq \left|\sum_{k=K_{j-1}+1}^{K_j} y_k x_k^{n_j}\right| - \left|\sum_{k=1}^{K_{j-1}} y_k x_k^{n_j}\right| - \left|\sum_{k=K_j+1}^{\infty} y_k x_k^{n_j}\right| \\
&\geq \sum_{k=K_{j-1}+1}^{K_j} |x_k^{n_j}| - \sum_{k=1}^{K_{j-1}} |x_k^{n_j}| - \sum_{k=K_j+1}^{\infty} |x_k^{n_j}| \\
&= \sum_{k=1}^{\infty} |x_k^{n_j}| - 2\sum_{k=1}^{K_{j-1}} |x_k^{n_j}| - 2\sum_{k=K_j+1}^{\infty} |x_k^{n_j}| \\
&> \|x^{n_j}\|_1 - \frac{4\delta}{5} \geq \frac{\delta}{5}.
\end{aligned}$$

Thus, we have $f(x^{n_j}) \nrightarrow 0$ as $j \to \infty$, contradicting the weak convergence of $(x^n)_{n\in\mathbb{N}}$ to 0.

b) By the previous point, every weakly convergent sequence in l_1 is norm convergent, and the converse is true in any normed space. Hence, if the statement was true that would mean that a set is weakly closed if and only if it is normed closed, i.e., the weak topology and the norm topology coincide on l_1. However, this is not true, as we have seen in Exercise (8).

Homework with solutions

(1) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the normed vector space $(X, \|\cdot\|)$ and let $x \in X$. Show that the following are equivalent.

(i) x_n weakly converges to x.

(ii) The sequence $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded and there exists a dense subset $D \subset X^*$ such that $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in D$.

Solution: (i)$\Longrightarrow$(ii): We have seen in the lecture that $x_n \to x$ weakly implies that $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded. Moreover, $x_n \to x$ weakly is equivalent to $f(x_n) \to f(x)$ for all $f \in X^*$; in particular it holds also for all $f \in D$ for any dense set $D \subseteq X^*$.

(ii)$\Longrightarrow$(i): We have to show that $f(x_n) \to f(x)$ for every $f \in X^*$. Let $f \in X^*$, and for every $\varepsilon > 0$, let $f_\varepsilon \in D$ be such that $\|f - f_\varepsilon\| < \varepsilon$. Then

$$\begin{aligned}|f(x_n) - f(x)| &= |f(x_n) - f_\varepsilon(x_n) + f_\varepsilon(x_n) - f_\varepsilon(x) + f_\varepsilon(x) - f(x)| \\ &\le |f(x_n) - f_\varepsilon(x_n)| + |f_\varepsilon(x_n) - f_\varepsilon(x)| + |f_\varepsilon(x) - f(x)| \\ &\le \|f - f_\varepsilon\|\,(M + \|x\|) + |f_\varepsilon(x_n) - f_\varepsilon(x)|,\end{aligned}$$

where $M := \sup_{n\in\mathbb{N}} \|x_n\|$, and hence

$$\limsup_{n\to+\infty} |f(x_n) - f(x)| \le \varepsilon(M + \|x\|) + \limsup_{n\to+\infty} |f_\varepsilon(x_n) - f_\varepsilon(x)| = \varepsilon(M + \|x\|).$$

Since this holds for every $\varepsilon > 0$, we get $f(x_n) \to f(x)$ as $n \to +\infty$.

(2) Let X_1 and X_2 be Banach spaces, and let $T : X_1 \to X_2$ be a linear transformation. Prove that if T is continuous relative to the weak topologies of X_1 and X_2, then T is bounded.

(Hint: Use the closed graph theorem.)

Solution: By the closed graph theorem, boundedness of T is equivalent to its graph $\Gamma(T)$ being closed. Thus, let $(x_n, Tx_n) \in \Gamma(T)$, $n \in \mathbb{N}$, be a sequence converging to some $(x, y) \in X_1 \times X_2$; we have to show that $(x, y) \in \Gamma(T)$, i.e., $y = Tx$. Since the weak topologies are at most as strong as the norm topologies, we see that x_n tends to x weakly, and Tx_n tends to y weakly. However, by assumption, T is continuous relative to the weak topologies, and thus $x_n \to x$ weakly implies $Tx_n \to Tx$ weakly. Since the weak topology is Hausdorff, this implies that $Tx = y$.

(3) Consider the differentiation operator $D : C^1([0,1]) \to C([0,1]),\ Df = f'$.

a) Prove that D has a closed graph if we equip both $C^1([0,1])$ and $C([0,1])$ with the $\|\cdot\|_\infty$ norm.

b) Conclude that $(C^1([0,1]), \|\cdot\|_\infty)$ is not a Banach space using the closed graph theorem.

Solution:

a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C^1([0,1])$ that converges to $f \in C^1([0,1])$ such that Df_n converges to some $g \in C([0,1])$. We have to prove that $Df = g$. To this end, note that

$$f_n(x) = f_n(0) + \int_0^x f_n(t)\,dt, \qquad \text{and let} \qquad \tilde{f}(x) = f(0) + \int_0^x g(t)\,dt,$$

for every $x \in [0,1]$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}\left\|f_n - \tilde{f}\right\|_\infty &= \sup_{x\in[0,1]} \left| f_n(0) - f(0) + \int_0^x (f_n'(t) - g(t))\,dt \right| \\ &\leq |f_n(0) - f(0)| + \sup_{x\in[0,1]} \int_0^x |f_n'(t) - g(t)|\,dt \\ &\leq \|f_n - f\|_\infty + \|f_n' - g\|_\infty \xrightarrow[n\to+\infty]{} 0.\end{aligned}$$

By the uniqueness of the limit, $f = \tilde{f}$, and, since g is continuous, $\tilde{f}$ is continuously differentiable with $D\tilde{f} = \tilde{f}' = g$. Hence, $g \in C^1([0,1])$, and $Df = g$.

b) We know $(C([0,1]), \|\cdot\|_\infty)$ is a Banach space. Assuming $(C^1([0,1]), \|\cdot\|_\infty)$ is a Banach space, the closed graph theorem would imply that D is continuous, a contradiction. Indeed, let $f_n(x) := \sin nx$ so that $\|f_n\|_\infty = 1$, $n \in \mathbb{N}$, but $\|Df_n\| = \sup_{x\in[0,1]} |n \cos nx| = n$, hence D is not bounded when both $C^1([0,1])$ and $C([0,1])$ are equipped with the maximum norm.