Introduction to Real Analysis

Real Numbers

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Field Axioms

$\mathbb{R}$ with two binary operations on $\mathbb{R}$ addition "$+$ "and multiplication "." from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the following properties

1. $a + b = b + a \quad \forall a, b \in \mathbb{R}$
   (commutative property of addition)
2. $(a + b) + c = a + (b + c) \quad \forall a, b, c \in \mathbb{R}$
   (associative property for addition)
3. $\exists 0 \in \mathbb{R} : \quad a + 0 = 0 + a \quad \forall a \in \mathbb{R}$
   (zero element)
4. $\forall a \in \mathbb{R} \quad \exists -a \in \mathbb{R} : a + (-a) = (-a) + a = 0$
   (additive inverse)

Field Axioms

1. $a.b = b.a \quad \forall a, b \in \mathbb{R}$
   (commutative property of multiplication)
2. $(a.b).c = a.(b.c) \quad \forall a, b, c \in \mathbb{R}$
   (associative property for multiplication)
3. $\exists 1 \neq 0 \in \mathbb{R} : a.1 = 1.a \quad \forall a \in \mathbb{R}$
   (unit element)
4. $\forall a \neq 0 \in \mathbb{R} \quad \exists a^{-1} \in \mathbb{R} : a.a^{-1} = a^{-1}.a = 1$
   (multipicative inverse)
5. $a.(b + c) = a.b + a.c \quad \forall a, b, c \in \mathbb{R}$
   (distributive property of multiplication over addition)

Order Axioms

Assume there is a subset $P \subset \mathbb{R}$ with the following properties

1. $\forall a \in \mathbb{R}$ either
   $a \in P$ or $a = 0$ or $-a \in P$
2. If $a, b \in P$ then $a + b \in P$ and $a.b \in P$

Triangle Inequality

Theorem

If $a, b \in \mathbb{R}$ then

1-

$$| a + b | \leq | a | + | b |$$

2-

$$|| a | - | b || \leq | a - b |$$

.

Bounded Sets

Definition

If $A \subset \mathbb{R}$

1. If there is $u \in \mathbb{R}$ such that
$$a \leq u \qquad \forall a \in A$$
the u is an upper bound of A, and A is bounded Above.
2. If there is $l \in \mathbb{R}$ such that
$$l \leq a \qquad \forall a \in A$$
the l is a lower bound of A, and A is bounded below.
3. A is bounded if it is bounded above and below.

Examples

Find an upper bound and a lower bound for the sets

1. $\{1, 2, 5\}$
2. $[2, 5)$
3. $\mathbb{Q}$
4. $\{\frac{1}{n} : n \in \mathbb{N}\}$

Supremum and Infimum

Definition

If $A \subset \mathbb{R}$, then an element $\beta \in \mathbb{R}$ is a least upper bound (supremum) if

1. β is an upper bound of A

$$a \leq \beta \qquad \forall a \in A$$

2. If there is an upper bound $u \in \mathbb{R}$ of A then

$$\beta \leq u$$

We use the notation

$$\beta = \sup A$$

Supremum and Infimum

Definition

If $A \subset \mathbb{R}$, then an element $\alpha \in \mathbb{R}$ is a greatest lower bound (infimum) if

1. α is a lower bound of A
2. If there is a lower bound $l \in \mathbb{R}$ of A then

$$\alpha \geq l$$

We use the notation

$$\alpha = \inf A$$

Examples

Find $\sup A, \inf A$

1. $A = \{-1, -2, 5, 9\}$
2. $A = [-1, 3)$

Maximum and Minimum

- If $\sup A \in A$ then $\sup A = \max A$
- If $\inf A \in A$ then $\inf A = \min A$

Examples

Find $\sup A, \inf A, \max A, \min A$

1. $A = \{1, 2, 5\}$
2. $A = [2, 5)$
3. $A = \mathbb{Q}$
4. $A = \{\frac{1}{n} : n \in \mathbb{N}\}$
5. $A = \{1 - \frac{1}{n}, n \in \mathbb{N}\}$

Examples

If A is any of the intervals $(a, b), (a, b], [a, b), [a, b]$ then

$$\sup A = b$$

$$\inf A = a$$

Completeness Axiom

Completeness Axiom

1. If $\phi \neq A \subset \mathbb{R}$ is bounded above then it has a least upper bound in $\mathbb{R}$
2. If $\phi \neq A \subset \mathbb{R}$ is bounded below then it has a greatest lower bound in $\mathbb{R}$

Completeness Axiom

Completeness Axiom

If we define $-A = \{-a : a \in A\}$ then A is bounded below iff $-A$ is bounded above and we have

$$\inf A = -\sup(-A)$$

$\sqrt{2}$

Theorem

There is no rational number x such that $x^2 = 2$

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Theorem

The set $\mathbb{N}$ is not bounded above.

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Archimedean Property

Archimedean Property

For every $x > 0$ there is $n \in \mathbb{N}$ such that $x > \frac{1}{n}$

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Corollary

for every $x \geq 0$ there is an $n \in \mathbb{N}$ such that $n - 1 \leq x < n$

.

Example

1. It

$$x \in \mathbb{R}, \quad x \leq \frac{1}{n} \quad \forall n \in \mathbb{N}$$

   then

2. if

$$x \in [0, \infty), \quad x \leq \frac{1}{n} \quad \forall n \in \mathbb{N}$$

   then

3. if

$$x \in \mathbb{R}^{+}, \quad x \leq \frac{1}{n} \quad \forall n \in \mathbb{N}$$

   then

Density of $\mathbb{Q}$

Theorem

Each open interval in $\mathbb{R}$ has a rational number.
If $x, y \in \mathbb{R},\ x < y$ there exist $r \in \mathbb{Q}$ such taht $x < r < y$

.

Density of $\mathbb{Q}^c$

Theorem

Each open interval in $\mathbb{R}$ has an irrational number.
If $x, y \in \mathbb{R},\ x < y$ there exist $t \in \mathbb{Q}^c$ such taht $x < t < y$

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Exercises

1. Determine $\sup A, \inf A, \max A, \min A$ where they exist;
   1. $A = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$
   2. $A = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$
   3. $A = \left\{x \in \mathbb{R} : x^2 - 4 < 0\right\}$
2. If b is an upper bound of A, prove that $b = \sup A$ if and only if for every $\varepsilon > 0$ there is an element $a \in A$ such that $a > b - \varepsilon$
3. If the sets A and B are bounded above and $A \subseteq B$, prove that $\sup A \leq \sup B$
4. Let A and B be bounded subsets of $\mathbb{R}$, and define

   $$A + B = \{a + b : a \in A, b \in B\}$$

   Prove that $\sup(A + B) = \sup A + \sup B$