## Introduction to Real Analysis Real Numbers

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Ibraheem Alolyan Real Analysis

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# Field Axioms

 $\mathbb{R}$  with two binary operations on  $\mathbb{R}$  addition "+" and multiplication "." from  $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$  with the following properties

• 
$$a + b = b + a \quad \forall a, b \in \mathbb{R}$$
  
(commutative property of addition)

② 
$$(a+b) + c = a + (b+c)$$
 ∀a, b, c ∈ ℝ  
(associative property for addition)

**3** 
$$\exists 0 \in \mathbb{R}$$
:  $a + 0 = 0 + a \quad \forall a \in \mathbb{R}$   
(zero element)

● 
$$\forall a \in \mathbb{R}$$
  $\exists -a \in \mathbb{R} : a + (-a) = (-a) + a = 0$   
(additive inverse)

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# Field Axioms

$$a.b = b.a \quad \forall a, b \in \mathbb{R}$$

(commutative property of multiplication)

② 
$$(a.b).c = a.(b.c)$$
 ∀ $a, b, c \in \mathbb{R}$   
(associative property for multiplication)

● 
$$\exists 1 \neq 0 \in \mathbb{R} : a.1 = 1.a \quad \forall a \in \mathbb{R}$$
  
( unit element)

● 
$$\forall a \neq 0 \in \mathbb{R}$$
  $\exists a^{-1} \in \mathbb{R} : a.a^{-1} = a^{-1}.a = 1$   
(multipicative inverse)

Image: A mathematical states and a mathem

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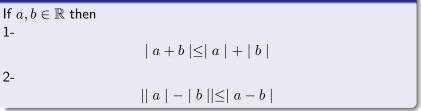
### Order Axioms

Assume there is a subset  $P \subset \mathbb{R}$  with the following properties

- $\forall a \in \mathbb{R}$  either  $a \in P$  or a = 0 or  $-a \in P$
- 2 If  $a, b \in P$  then  $a + b \in P$  and  $a.b \in P$

# Triangle Inequality

#### Theorem



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## **Completeness** Axiom

#### Definition

- $\mathsf{If}\ A \subset \mathbb{R}$ 
  - **1** If there is  $u \in \mathbb{R}$  such that

$$a \le u \qquad \forall a \in A$$

the  $\boldsymbol{u}$  is an upper bound of  $\boldsymbol{A}\text{,}$  and  $\boldsymbol{A}$  is bounded Above.

2 If there is  $l \in \mathbb{R}$  such that

$$l \le a \qquad \forall a \in A$$

the l is a lower bound of  $A, \, {\rm and} \, {\rm A}$  is bounded below.

 $\bigcirc$  A is bounded if it is bounded above and below.

# Supremum and Infimum

### Definition

If  $A \subset \mathbb{R},$  then an element  $\beta \in \mathbb{R}$  is a least upper bound (supremum) if

• 
$$\beta$$
 is an upper bound of  $A$ 

$$a \leq \beta \qquad \forall a \in A$$

2 If there is an upper bound  $u \in \mathbb{R}$  of A then

$$\beta \leq u$$

We use the notation

$$\beta = \sup A$$

# Supremum and Infimum

### Definition

If  $A \subset \mathbb{R}$ , then an element  $\alpha \in \mathbb{R}$  is a greatest lower bound (infimum) if

- $\textbf{0} \ \alpha \text{ is a lower bound of } A$
- **2** If there is a lower bound  $l \in \mathbb{R}$  of A then

$$\alpha \ge l$$

We use the notation

$$\alpha = \inf A$$

## Maximum and Minimum

- If  $\sup A \in A$  then  $\sup A = \max A$
- If  $\inf A \in A$  then  $\inf A = \min A$

## Examples

\$\{1,2,5\}\$
\$\[[2,5)\$]
\$\[Q\$\$
\$\{\frac{1}{n}:n\in\mathbb{N}\$}\$
\$\{1-\frac{1}{n},n\in\mathbb{N}\$}\$

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## Examples

### If A is any of the intervals (a, b), (a, b], [a, b), [a, b] then

 $\sup A = b$ 

 $\inf A = a$ 

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Image: A matrix and a matrix

### **Completeness** Axiom

#### Completeness Axiom

- If  $\phi \neq A \subset \mathbb{R}$  is bounded above then it has a least upper bound in  $\mathbb{R}$
- ② If  $\phi \neq A \subset \mathbb{R}$  is bounded below then it has a greatest lower bound in  $\mathbb{R}$

### **Completeness Axiom**

#### Completeness Axiom

If we define  $-A = \{-a : a \in A\}$  then A is bounded below iff -A is bounded above and we have

$$\inf A = -\sup(-A)$$



There is no rational number x such that  $x^2 = 2$ 

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The set  $\mathbb{N}$  is not bounded above.

Archimedean Property

For every x > 0 there is  $n \in \mathbb{N}$  such that  $x > \frac{1}{n}$ 

#### Corollary

for every  $x \geq 0$  there is an  $n \in \mathbb{N}$  such that  $n-1 \leq x < n$ 

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### Exercise

1 lt	$x \in \mathbb{R},  x \leq \frac{1}{n}  \forall n \in \mathbb{N}$
then	
if	$x \in [0,\infty),  x \leq \frac{1}{n}  \forall n \in \mathbb{N}$
then	
3 if	$x \in \mathbb{R}^+,  x \leq \frac{1}{n}  \forall n \in \mathbb{N}$
then	

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Each open interval in  $\mathbb{R}$  has a rational number. If  $x, y \in \mathbb{R}$ , x < y there exist  $r \in \mathbb{Q}$  such that x < r < y



Each open interval in  $\mathbb{R}$  has an irrational number. If  $x, y \in \mathbb{R}$ , x < y there exist  $t \in \mathbb{Q}^c$  such that x < t < y

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