Show that the rate of convergence of Newton's method at the root $\alpha = 1$ of the equation $(x-1)^2 \sin x = 0$ is linear. Use quadratic convergence method to find x_2 using $x_0 = 1.5$. Compute the relative error.

Question 3: Find the values of a, b and c such that the iterative scheme

$$x_{n+1} = ax_n + \frac{bN}{x_n^2} + \frac{cN^2}{x_n^5}, \qquad n \geq 0,$$

converges at least cubically to $\alpha = N^{\frac{1}{3}}$. Use this scheme to find second approximation of $(8)^{\frac{1}{3}}$ when $x_0 = 1.8$. [6 Marks]

Find the first approximation for the nonlinear system

$$\begin{aligned} y^2(1-x) &= x^3 \\ x^2 + y^2 &= 1 \end{aligned}$$

using Newton's method, starting with initial approximation $(x_0, y_0)^T = (1, 1)^T$.

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$$f(x) = (x-1)^2 \sin x \quad \text{and} \quad f'(x) = 2(x-1)\sin x + (x-1)^2 \cos x,$$

and $f'(1) = 0$, gives that $\alpha = 1$ is the multiple root. Using Newton's iterative formula, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{((x_n-1)^2 \sin x_n)}{(2(x_n-1)\sin x_n + (x_n-1)^2 \cos x_n)} = x_n - \frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)},$$

for $n \geq 0$. The fixed point form of the developed Newton's formula is

$$x_{n+1} = g(x_n) = x_n - \frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)}.$$

Then

$$g(x) = x - \frac{((x-1)\sin x)}{(2\sin x + (x-1)\cos x)},$$

and

$$g'(x) = 1 - \frac{(2\sin x + (x-1)\cos x)(\sin x + (x-1)\cos x) - ((x-1)\sin x)(3\cos x - (x-1)\sin x)}{(2\sin x + (x-1)\cos x)^2}.$$

Thus

$$g'(1) = 1 - \frac{2(\sin 1)^2}{4(\sin 1)^2} = \frac{1}{2} \neq 0,$$

and so the Newton's method converges linearly. The quadratic convergent method for multiple root is modified Newton's method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad n \geq 0,$$

where m is the order of multiplicity of the zero of the function. To find m, we check that

$$f''(x) = 2\sin x + 4(x-1)\cos x - (x-1)^2 \sin x, \quad \text{and} \quad f''(1) = 2\sin 1 \neq 0,$$

so $m = 2$. Thus

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} = x_n - 2\frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)}, \qquad n \geq 0.$$

Now using initial approximation $x_0 = 1.5$, we have the following two approximations

$$x_1 = x_0 - 2\frac{((x_0-1)\sin x_0)}{(2\sin x_0 + (x_0-1)\cos x_0)} = 1.0087, \quad x_2 = x_1 - 2\frac{((x_1-1)\sin x_1)}{(2\sin x_1 + (x_1-1)\cos x_1)} = 1.0000,$$

Solution. Given the iterative scheme converges at least cubically means $g' = g'' = 0$ at $\alpha = N^{\frac{1}{3}}$. Let

$$g(x) = ax + \frac{bN}{x^2} + \frac{cN^2}{x^5}, \quad g(N^{\frac{1}{3}}) = 1 = a + b + c,$$

$$g'(x) = a - \frac{2bN}{x^3} - \frac{5cN^2}{x^6}, \quad g'(N^{\frac{1}{3}}) = 0 = a - 2b - 5c,$$

$$g''(x) = 0 + \frac{6bN}{x^4} + \frac{30cN^2}{x^7}, \quad g''(N^{\frac{1}{3}}) = 0 = 3b + 15c,$$

Solving these three equations for unknowns a, b and c, we obtain $a = b = \frac{5}{9}$ and $c = -\frac{1}{9}$. Thus

$$x_{n+1} = \frac{5x_n}{9} + \frac{5N}{9x_n^2} - \frac{N^2}{9x_n^5}, \qquad n \geq 0.$$

Using $x_0 = 1.8,\ N = 8$, we obtain

$$x_1 = \frac{5x_0}{9} + \frac{5N}{9x_0^2} - \frac{N^2}{9x_0^5} = 1.9954,$$

and

$$x_2 = \frac{5x_1}{9} + \frac{5N}{9x_1^2} - \frac{N^2}{9x_1^5} = 2.0000,$$

the required second approximation.
The absolute error can be obtained as

$$|(8)^{1/3} - 2.0000| = |2.0000 - 2.0000| = 0.0000 \quad \text{up to 4 dp.}$$

Question 5: Find the first approximation for the nonlinear system

$$\begin{aligned} y^2(1-x) &= x^3 \\ x^2+y^2 &= 1 \end{aligned}$$

using Newton's method, starting with initial approximation $(x_0, y_0)^T = (1, 1)^T$.

Solution. Given

$$\begin{aligned} f_1(x,y) &= y^2(1-x)-x^3, \quad f_{1_x} = -y^2-3x^2, \quad f_{1_y} = 2y(1-x), \\ f_2(x,y) &= x^2+y^2-1, \quad f_{2_x} = 2x, \quad f_{2_y} = 2y. \end{aligned}$$

At the given initial approximation $x_0 = 1$ and $y_0 = 1$, we have

$$\begin{aligned} f_1(1,-1) &= -1, \quad \frac{\partial f_1}{\partial x} = f_{1_x} = -4, \quad \frac{\partial f_1}{\partial y} = f_{1_y} = 0, \\ f_2(1,1) &= 1, \quad \frac{\partial f_1}{\partial x} = f_{2_x} = 2, \quad \frac{\partial f_2}{\partial y} = f_{2_y} = 2. \end{aligned}$$

The Jacobian matrix J at the given initial approximation can be calculated as

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad J^{-1} = \frac{1}{-8}\begin{pmatrix} 2 & 0 \\ -2 & -4 \end{pmatrix},$$

is the inverse of the Jacobian matrix.Now to find the first approximation we have to solve the following equation

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{-8}\begin{pmatrix} 2 & 0 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.75 \end{pmatrix},$$

the required first approximation. •