King Saud University	College of Sciences	Department of Mathematics
Quiz 1	CSC 281	Semester 1 (1448)

Name:..... Student Number:.....

The examination contain 2 pages.

Question 1 (2 marks)

Use rules of inference to show that if

$$\forall x,\ (P(x) \vee Q(x)) \qquad \text{and} \qquad \forall x,\ ((P(x) \wedge \neg Q(x)) \to R(x))$$

are true, then $\forall x,\ (\neg R(x) \to Q(x))$ is true.

Solution. Let a be an arbitrary element of the domain. By universal instantiation,

$$P(a) \vee Q(a), \qquad (P(a) \wedge \neg Q(a)) \to R(a).$$

Assume $\neg R(a)$. To prove $Q(a)$, suppose, for a contradiction, that $\neg Q(a)$ is true. The rules of inference give the following steps:

Step	Statement	Justification
1	$P(a) \vee Q(a)$	Universal instantiation of the first premise.
2	$(P(a) \wedge \neg Q(a)) \to R(a)$	Universal instantiation of the second premise.
3	$\neg R(a)$	Assumption for proving the conditional.
4	$\neg Q(a)$	Temporary assumption for contradiction.
5	$P(a)$	Disjunctive syllogism, using steps 1 and 4.
6	$P(a) \wedge \neg Q(a)$	Conjunction, using steps 5 and 4.
7	$R(a)$	Modus ponens, using steps 2 and 6.

Steps 3 and 7 contradict one another. Therefore the temporary assumption $\neg Q(a)$ is false, and $Q(a)$ is true. We have proved

$$\neg R(a) \to Q(a).$$

Since a was arbitrary and no special property of a was assumed, universal generalization gives

$$\boxed{\forall x,\ (\neg R(x) \to Q(x)).}$$

Question 2 (1.5 marks)

Let $n \in \mathbb{Z}$. Show that: If $n^2 - 4n + 7$ is even, then n is odd.

Solution (proof by contraposition).

The contrapositive is: if n is not odd, then $n^2 - 4n + 7$ is not even.

Since $n \in \mathbb{Z}$, we also have $n^2 - 4n + 7 \in \mathbb{Z}$. Every integer is either even or odd, but not both. Therefore, the contrapositive can be equivalently stated as follows: if n is even, then $n^2 - 4n + 7$ is odd.

Suppose that n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Consequently,

$$\begin{aligned} n^2 - 4n + 7 &= (2k)^2 - 4(2k) + 7 \\ &= 4k^2 - 8k + 7 \\ &= 2(2k^2 - 4k + 3) + 1. \end{aligned}$$

Since $2k^2 - 4k + 3 \in \mathbb{Z}$, this expression is odd. Thus the contrapositive is true, and hence

$$\boxed{\forall n \in \mathbb{Z};\ \text{If } n^2 - 4n + 7 \text{ is even, then } n \text{ is odd.}}$$

Question 3 (1.5 marks)

Let $n \in \mathbb{Z}$. Show that: $3 \mid n(n^2 - 1)$.

Solution (proof by cases). First factor the expression:

$$n(n^2 - 1) = (n - 1)n(n + 1).$$

By the division algorithm, every integer n has exactly one of the forms $3k$, $3k + 1$, or $3k + 2$, where $k \in \mathbb{Z}$.

- If $n = 3k$, then $3 \mid n$.
- If $n = 3k + 1$, then $n - 1 = 3k$, so $3 \mid (n - 1)$.
- If $n = 3k + 2$, then $n + 1 = 3(k + 1)$, so $3 \mid (n + 1)$.

In every case, one of the three factors $(n - 1)$, n, and $(n + 1)$ is a multiple of 3. Their product is therefore a multiple of 3. Hence

$$\boxed{3 \mid n(n^2 - 1) \quad \text{for every } n \in \mathbb{Z}.}$$