FINAL EXAMINATION, SEMESTER I, 2025-26

DEPT. MATH., COLLEGE OF SCIENCE, KSU

MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Note: You are NOT allowed to use calculator

Q1. [4+3+2=9] (a) (i) Find inverse of the matrix A by the elementary matrix method

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

(ii) Using the inverse of A, solve the following system of linear equations:

$$3x - y + 2z = 0$$
$$2x - 3y + z = 1$$
$$x - 2y + z = 2.$$

(iii) Using A^{-1} (as you obtained in (i)), find $adj(A)$ (adjoint of the matrix A.)

(b) If

$$B = \begin{bmatrix} \sqrt{5}\lambda - 1 & -2 \\ -2 & \sqrt{5}\lambda + 1 \end{bmatrix},$$

find all values of λ such that $\det(B) = \det\left(\begin{bmatrix} -2 \\ \lambda \end{bmatrix}\begin{bmatrix} -1 & \lambda^2 \end{bmatrix}\right)$.

(c) Using Cramer's Rule find the value of z for the system of linear equations:

$x + y + z = 6 \quad x + y + 2z = 9 \quad 2x + y + z = 7.$

Q2. [2+2+3=7] (a) Find the volume of the parallelepiped (box) having $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{c} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.

(b) Find distance between the two parallel planes given by $3x - y + 2z - 6 = 0$ and $6x - 2y + 4z + 4 = 0$.

(c) Find the equation of the plane containing the points $P(2, 1, 1)$, $Q(0, 4, 1)$, and $R(-2, 1, 4)$.

Q3. [3+3+3=9] (a) Describe the curve defined by the vector valued function $\mathbf{r}(t) = \langle 1, 2\cos t, 2\sin t\rangle$; what is its domain? Find its curvature at $t = -\pi$.

(b) Find parametric equations for the tangent line to the curve $C : x = 8t + 2,\ y = 2t^3 - 1,\ z = -5t^2 + 3$ at the point $(10, 1, -2)$.

(c) Find the tangential and normal components of acceleration of a moving particle with position vector given by $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$.

Q4. [3+3+3+3+3=15] (a) If $w = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, show that $(\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2 = (\frac{\partial w}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial w}{\partial \theta})^2$

(b) Let $f(x, y) = x^3y^2$. Find the directional derivatives of f at the point $P(-1, 2)$ in the direction of the vector $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$.

(c) Find the equations of the tangent plane and the normal line to the surface $z = \sin x \cos y$ at $(\frac{\pi}{2}, \pi, -1)$.

(d) If $f(x, y) = -x^3 + 4xy - 2y^2 + 1$, find the local extrema and saddle points of f, if any.

(e) Use Lagrange multiplier to find the minimum value of $f(x, y, z) = 2x^2 + y^2 + 3z^2$ subject to the constraint $2x - 3y - 4z = 49$.

Q1. [4+3+2=9] (a) (i) Starting with $[A|I]$ form, and applying elementary matrix method, we arrive at the form $[I|A^{-1}]$, where

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & \frac{-5}{4} \\ \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{-5}{4} & \frac{7}{4} \end{bmatrix}$$

(ii) Using A^{-1} from (i), and $X = A^{-1}B$, we get $x = -\frac{7}{4}$, $y = -\frac{3}{4}$ and $z = \frac{9}{4}$.

(iii) Since $det(A) = -4$, and $adj(A) = \det(A)A^{-1}$, we obtain

$$adj(A) = \begin{bmatrix} -1 & -3 & 5 \\ -1 & 1 & 1 \\ -1 & 5 & -7 \end{bmatrix}$$

(b) From the given equation, we get $5\lambda^2 - 1 - 4 = 0$ which implies $\lambda = \pm 1$.

(c) Let A be the coefficient matrix of the given system, then $det(A) = 1$ and $z = \frac{det(A_3)}{\det(A)} = 3$.

Q2. [2+2+3=7] (a) We know volume, $V = |\mathbf{a}.(\mathbf{b} \times \mathbf{c})|$. Upon using this, we get $V = 36$.

(b) To find the distance between the planes, choose a point in the first plane, say $(x_0, y_0, z_0) = (2, 0, 0)$. Then from the second plane, we can determine $a = 6, b = -2, c = 4$, and $d = 4$, and conclude the distance $D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{16}{\sqrt{56}}$.

(c) Here $\mathbf{PQ} = \langle -2, 3, 0\rangle$ and $\mathbf{PR} = \langle -4, 0, 3\rangle$. Then $\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \langle 9, 6, 12\rangle$ is normal to the given plane. Using the direction numbers for $\mathbf{n}$ and the point $P(2, 1, 1)$, the required equation of the plane is $3x + 2y + 4z - 12 = 0$.

Q3. [3+3+3=9] (a) (Part I) We have $x = 1$, $y = 2\cos t$, and $z = 2\sin t$ implying $\cos t = \frac{y}{2}$ and $\sin t = \frac{z}{2}$ and hence $\frac{y^2}{4} + \frac{z^2}{4} = 1$. (Part II) The domain of $\mathbf{r}(t)$ is $\Re$. (Part III) The curvature $\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{\|\mathbf{r}'(t)\|^3} = \frac{4}{2^3} = \frac{1}{2}$. Hence the curvature at $t = -\pi$ is $\kappa = \frac{1}{2}$.

(b) Here $\mathbf{r}(t) = \langle 8t + 2, 2t^3 - 1, -5t^2 + 3\rangle$, and at $(10, 1, -2)$ one obtains $t = 1$. We have $\mathbf{r}'(t) = \langle 8, 6t^2, -10t\rangle$, and $\mathbf{r}'(1) = \langle 8, 6, -10\rangle$ is the direction vector of the tangent line. Hence, then parametric equations for the tangent line is given by $x = 10 + 8t$, $y = 1 + 6t$ $z = -2 - 10t$.

(c) We get $\mathbf{r}'(t) = \langle 3, -1, 2t\rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{10 + 4t^2}$.

Then the tangential component is $a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\mathbf{r}'(t)} = \frac{4t}{\sqrt{10+4t^2}}$, and the normal component of acceleration is $a_N = \frac{\|\mathbf{r}''(t) \times \mathbf{r}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{4+36}}{\sqrt{10+4t^2}} = \frac{\sqrt{40}}{\sqrt{10+4t^2}}$.

Q4. [3+3+3+3+3=15] (a) We have $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$ and $\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$. Then upon using $\sin^2\theta + \cos^2\theta = 1$, one obtains: $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$.

(b) The unit vector $\mathbf{u} = \langle \frac{4}{5}, -\frac{3}{5}\rangle$. Since $f_x(x, y) = 3x^2y^2$ and $f_y(x, y) = 2x^3y$ $\Rightarrow D_{\mathbf{u}}f(-1, 2) = 3(-2)^2(2)^2(\frac{4}{5}) + 2(-1)^3(2)(-\frac{3}{5}) = 12$.

(c) Given $z = f(x, y) = \sin x \cos y$. We get $\frac{\partial f}{\partial x} = \cos x \cos y$ and $\frac{\partial f}{\partial y} = -\sin x \sin y$. The equation of the tangent plane to the surface $z = f(x, y)$ at $(\frac{\pi}{2}, \pi, -1)$ is $z = -1$; and the parametric equations for the normal lines are $x = \frac{\pi}{2}$, $y = -\pi$, $z = -1 + t$.

(d) We have $f_x(x, y) = -3x^2 + 4y$ and $f_y(x, y) = 4x - 4y$. The critical points are: $(0, 0)$ and $(\frac{4}{3}, \frac{4}{3})$. We also have $f_{xx}(x, y) = -6x$, $f_{xy}(x, y) = 4$, and $f_{yy}(x, y) = -4$. It follows for the critical point $(0, 0)$, $D(0, 0) = -16 < 0$ in which case one obtains saddle point, and for the critical point $(\frac{4}{3}, \frac{4}{3}) = 16 > 0$, one obtains local maximum since $f_{xx}(\frac{4}{3}, \frac{4}{3}) = -16 < 0$.

(e) Let $g(x, y, z) = 2x - 3y - 4z = 49$. By Lagrange multiplier, we get $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ from which we obtain: $4x = 2\lambda$, $2y = -3\lambda$ and $6z = -4\lambda$ and $2x - 3y - 4z = 49$. Solving these equations, we get $x = 3$, $y = -9$, and $z = -4$. Therefore, the minimum value of f is $f(3, -9, -4) = 147$.