Show that the best iterative formula for computing the approximation of the root $\alpha = 0$ of the equation $x^3 e^{2x} = 0$ is

$$x_{n+1} = x_n - \frac{3x_n}{(3 + 2x_n)}, \qquad n \geq 0.$$

Use it to find the absolute error $|\alpha - x_2|$ using $x_0 = 0.1$. Show that the given iterative formula is converges quadratically to a root $\alpha = 0$.

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$f(x) = x^3 e^{2x} \to f'(x) = 3x^2 e^{2x} + 2e^{2x} \cdot x^3$

$f'(0) = 0 \to f''(x) = 6x e^{2x} + 3x^2 \cdot 2e^{2x} + 4e^{2x} x^3 + 2e^{2x} \cdot 3x^2$

$f''(0) = 0 \to$

$f'''(x) = 6e^{2x} + 6x \cdot 2e^{2x} + 12x e^{2x} + 6x^2 \cdot 2e^{2x} + 8e^{2x} \cdot x^3 + 4e^{2x} \cdot 3x^2$
$+ 4e^{2x} \cdot 3x^2 + 2e^{2x} \cdot 6x$

$f'''(0) = 6 \neq 0 \Rightarrow m = 3$

The Best iterative formula is Modified Newton:

$$x_{n+1} = x_n - \frac{m f(x_n)}{f'(x_n)} = x_n - \frac{3 \cdot x_n^3 e^{2x_n}}{3x_n^2 e^{2x_n} + 2e^{2x_n} x_n^3} , \quad n \geq 0$$

$$= x_n - \frac{3x_n^{\cancel{3}} \cancel{e^{2x_n}}}{\cancel{x_n^2} \cancel{e^{2x_n}} (3 + 2x_n)}$$

$$x_{n+1} = x_n - \frac{3x_n}{3 + 2x_n} , \quad n \geq 0$$

$n = 0 \to x_1 = x_0 - \frac{3x_0}{3 + 2x_0} = 0.1 - \frac{3(0.1)}{3 + 2(0.1)} = 0.00625$

$n = 1 \to x_2 = x_1 - \frac{3x_1}{3 + 2x_1} = 0.0000259$

Absolute error $= |\alpha - x_2| = |0 - 0.0000259| = 0.0000259$

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$$g(x) = x - \frac{3x}{3+2x}$$

$g(0) = 0 \Rightarrow 0$ is fixed point of g

$$g'(x) = 1 - \frac{(3+2x)\cdot 3 - 3x\cdot 2}{(3+2x)^2} = 1 - \frac{9}{(3+2x)^2}$$

$$g'(\alpha) = g'(0) = 1 - \frac{9}{(3+0)^2} = 1 - 1 = 0$$

$$g'(x) = 1 - 9(3+2x)^{-2}$$

$$g''(x) = 0 + 18(3+2x)^{-3}\cdot 2$$

$$= \frac{36}{(3+2x)^3}$$

$$g''(\alpha) = g''(0) = \frac{36}{(3+0)^3} = \frac{36}{27} = \frac{4}{3} \neq 0$$

$\Rightarrow$ Rate of convergence is $\underline{\underline{2}}$ (quadratic)