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ORIGINAL RESEARCH

Quasi Jordan algebras with involution

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Abstract We initiate a study of involutions in the setting of complex quasi Jordan algebras and discuss the notions of self-adjoint and unitary elements; besides other results, we also obtain a Russo-Dye type theorem for unital involutive split quasi Jordan Banach algebras.

Keywords Jordan algebra · Jordan Banach algebra · Dialgebra · Leibniz algebra · Quasi Jordan algebra · Quasi Jordan Banach algebra · Split quasi Jordan algebra · Jordan part · Zero part · Involution · Self-adjoint element · Unitary element

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1 Introduction and Preliminaries

A Jordan algebra is a non-associative algebra $\mathcal{J}$ over a field of characteristic $\neq 2$, where the product $x \circ y$ is commutative satisfying the Jordan identity: $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$. It is well known that the notion of Jordan algebras was appeared in 1930s as a result of the quantum mechanical formalism (cf. [10, 11]); these algebras and related structures have been extensively studied by a large number of mathematicians. In 1960s, people began studying Jordan structures from the functional analytic point of view (cf. [19, 21]); interesting theories of Jordan Banach algebras, JB-algebras, JB^*-algebras and JB^*-triples have been developed, which closely resemble the C^*-algebra theory and have found surprisingly important applications in a wide range of mathematical disciplines (cf. [2, 3, 5–9, 13, 22, 23, 27]).

In 2008, R. Velásquez and R. Felipe [24] introduced a non-commutative generalization of Jordan algebras, called quasi Jordan algebras, where the commutativity of the product is replaced by a quasi commutative identity and a weaker form of the Jordan identity is retained (see below). Recently in [1], with the hope that quasi Jordan algebra analogue of Jordan Banach algebras may provide an extended and better mathematical foundation for some important areas, the present authors initiated a study of the quasi Jordan

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Banach algebras. This new class of algebras properly includes all Jordan Banach algebras; hence, all JB^*-algebras and all C^*-algebras. In this article, we investigate the notions of involutions, self-adjoint and unitary elements in the setting of complex quasi Jordan algebras and quasi Jordan Banach algebras. Among other related results, we also obtain a Russo-Dye type theorem for involutive split quasi Jordan Banach algebras.

Leibniz Algebras and Dialgebras. For any Jordan algebra $\mathcal{J}$, there is a Lie algebra $\mathcal{L}_{\mathcal{J}}$ such that $\mathcal{J}$ is a linear subspace of $\mathcal{L}_{\mathcal{J}}$ and the Jordan product in $\mathcal{J}$ can be expressed in terms of the Lie bracket in $\mathcal{L}_{\mathcal{J}}$; moreover, the universal enveloping algebra of a Lie algebra is an associative algebra (cf. [3, 12, 14, 20]). In 1993, J. Loday introduced a non-commutative generalization of Lie algebras, called Leibniz algebras. A Leibniz algebra over a field K is a K-vector space L equipped with a bilinear product $[\ ,\ ]$, called Leibniz bracket, satisfying the Leibniz identity: $[[x, y], z] = [x, [y, z]] + [[x, z], y]$ (cf. [15, 17]); of course, the algebra L becomes a Lie algebra if the Leibniz bracket is skew-symmetric. Later on, Loday demonstrated that the relationship between Lie algebras and associative algebras is analogous to the relationship between Leibniz algebras and the so called dialgebras (cf. [16]): a dialgebra over a field K is a K-vector space D equipped with bilinear associative maps $\dashv, \vdash : D \times D \to D$ satisfying $x \dashv (y \vdash z) = x \dashv (y \dashv z)$; $(x \vdash y) \dashv z = x \vdash (y \dashv z)$ and $(x \dashv y) \vdash z = (x \vdash y) \vdash z$. Clearly, any linear algebra with associative product xy is a dialgebra, where $x \dashv y := xy$ and $x \vdash y := xy$. Any dialgebra $(D, \dashv, \vdash)$ becomes a Leibniz algebra under the Leibniz bracket $[x, y] := x \dashv y - y \vdash x$, and the universal enveloping algebra of a Leibniz algebra is a dialgebra (cf. [16, 17]).

Quasi Jordan Algebras. In [24], R. Velásquez and R. Felipe introduced the notion of quasi Jordan algebras as follows: a quasi Jordan algebra is a vector space $\mathfrak{I}$ over a field K of characteristic $\neq 2$ equipped with a bilinear map $\lhd : \mathfrak{I} \times \mathfrak{I} \to \mathfrak{I}$, called the quasi Jordan product, satisfying $x \lhd (y \lhd z) = x \lhd (z \lhd y)$ (the right commutativity) and $(y \lhd x^2) \lhd x = (y \lhd x) \lhd x^2$ (the right Jordan identity). Thus, any Jordan algebra is a quasi Jordan algebra.

Any dialgebra $(D, \dashv, \vdash)$ over field K of characteristic $\neq 2$ is a quasi Jordan algebra under the quasi Jordan product $x \lhd y := \frac{1}{2}(x \dashv y + y \vdash x)$. This induced algebra is denoted by D^+; called a plus quasi Jordan algebra. Quasi Jordan algebras have relation with Leibniz algebras similar to the above mentioned relationship between Jordan algebras and Lie algebras (cf. [26]).

An element e in a quasi Jordan algebra $\mathfrak{I}$ is called a right unit if $x \lhd e = x,\ \forall x \in \mathfrak{I}$. If there is an element l in a quasi Jordan algebra $\mathfrak{I}$ satisfying $l \lhd x = x,\ \forall x \in \mathfrak{I}$, called a left unit, then $\mathfrak{I}$ is necessarily commutative and so it is a Jordan algebra. Henceforth, by a unit in a quasi Jordan algebra we would mean a right unit unless explicitly stated otherwise. If a dialgebra D has a bar-unit e (that is, an element e satisfying $x \dashv e = x = e \vdash x,\ \forall x \in D$) then $x \lhd e = x,\ \forall x \in D$ and so e is a (right) unit of D^+; notice that such a bar-unit may not be a left unit since $e \lhd x = \frac{1}{2}(e \dashv x + x \vdash e)$ may not be equal to x. A dialgebra may have more than one bar-units and so the same is true for units in a quasi Jordan algebra; in fact, a quasi Jordan algebra may have infinitely many units (cf. [25, 26]).

Any quasi Jordan algebra $\mathfrak{I}$ includes

$$\mathfrak{I}^{\mathfrak{ann}} = \mathfrak{span}\{\mathfrak{x} \lhd \mathfrak{y} - \mathfrak{y} \lhd \mathfrak{x} \ \ \mathfrak{x}, \mathfrak{y} \in \mathfrak{I}\}$$

and

$$Z(\mathfrak{I}) := \{z \in \mathfrak{I} \ \ \mathfrak{x} \lhd \mathfrak{z} = , \forall \mathfrak{x} \in \mathfrak{I}\},$$

respectively called annihilator and zero part of $\mathfrak{I}$. It follows from the right commutativity that $\mathfrak{I}^{\mathfrak{ann}} \subseteq \mathfrak{Z}(\mathfrak{I})$; and that the quasi Jordan algebra $\mathfrak{I}$ is a Jordan algebra if and only if $\mathfrak{I}^{\mathfrak{ann}} = \{\}$. Further, if a quasi Jordan algebra $\mathfrak{I}$ has a unit e then $\mathfrak{I}^{\mathfrak{ann}}$ and $Z(\mathfrak{I})$ are two-sided ideals of $\mathfrak{I}$, $\mathfrak{I}^{\mathfrak{ann}} = \{\mathfrak{x} \in \mathfrak{I} \ \ \mathfrak{e} \lhd \mathfrak{x} = \} = \mathfrak{Z}(\mathfrak{I})$ and $U(\mathfrak{I}) = \{x + e : x \in Z(\mathfrak{I})\}$, where $U(\mathfrak{I})$ denotes the set of all units in $\mathfrak{I}$ (cf. [25]).

Split Quasi Jordan Algebras. Investigating unitization of quasi Jordan algebras, Velásquez and Felipe [25] introduced the class of split quasi Jordan algebras: a quasi Jordan algebra $\mathfrak{I}$ is called a split (more precisely, split over I) if I is an ideal in $\mathfrak{I}$ with $\mathfrak{I}^{\mathfrak{ann}} \subseteq \mathfrak{I} \subseteq \mathfrak{Z}(\mathfrak{I})$ and there exists a subalgebra J of $\mathfrak{I}$ such that $\mathfrak{I} = \mathfrak{J} \oplus \mathfrak{I}$ (the direct sum of J and I). In such a case, the subalgebra J is necessarily commutative because $x \lhd y - y \lhd x \in \mathfrak{I}^{\mathfrak{ann}} \cap \mathfrak{J} \subseteq \mathfrak{I} \cap \mathfrak{J} = \{\}$ for all $x, y \in J$; hence, J is a Jordan algebra. Clearly, a quasi Jordan algebra $\mathfrak{I}$ with a unit is a split quasi Jordan algebra if and only if $\mathfrak{I} = \mathfrak{J} \oplus \mathfrak{Z}(\mathfrak{I})$ for some subalgebra J of $\mathfrak{I}$; the algebra J is called the Jordan part of $\mathfrak{I}$. In this case, every element $x \in \mathfrak{I}$ has a unique representation $x = x_{_J} + x_{_Z}$ with $x_{_J} \in J$ and $x_{_Z} \in Z(\mathfrak{I})$, respectively called the Jordan part and the zero part of x. For any unital quasi Jordan algebra $\mathfrak{I}$, $\mathfrak{I} = \{(\mathfrak{x}, \mathfrak{R}_{\mathfrak{y}}) \ \ \mathfrak{x}, \mathfrak{y} \in \mathfrak{I}\}$ with the sum

$(a, R_b) + (c, R_d) := (a + c, R_{b+d})$, scalar multiplication $\lambda(a, R_b) := (\lambda a, R_{\lambda b})$ and product $(a, R_b) \lhd (c, R_d) := (a \lhd d, R_{b \lhd d})$ is a split quasi Jordan algebra; the map $x \longmapsto \varphi(x) := (x, R_x)$ is an embedding of $\mathfrak{I}$ into $\mathfrak{I}$; here, R_z denotes the usual right multiplication operator on $\mathfrak{I}$ (cf. [25]). It is easily seen that $Z(\mathfrak{I}) = \mathfrak{I} \times \{\mathfrak{R}\}$ and $\{0\} \times R(\mathfrak{I})$ are respectively the zero and Jordan parts of $\mathfrak{I}$, where $R(\mathfrak{I}) := \{R_x : x \in \mathfrak{I}\}$. The embedding φ preserves the units: clearly, $(a, R_b) \lhd (e, R_e) = (a \lhd e, R_{b \lhd e}) = (a, R_b),\ \forall (a, R_b) \in \mathfrak{I}$; so that (e, R_e) is a unit in $\mathfrak{I}$ whenever e is a unit in $\mathfrak{I}$; in fact, (x, R_e) is a unit in $\mathfrak{I},\ \forall \mathfrak{x} \in \mathfrak{I}$, and $(0, R_e)$ is the only unit in the Jordan part of $\mathfrak{I}$. We observe that $R(\mathfrak{I})$ is a quasi Jordan algebra with addition $R_x + R_y := R_{x+y}$, scalar multiplication $\lambda R_x := R_{\lambda x}$ and product $R_x \lhd R_y := R_{x \lhd y}$. Moreover, $\mathfrak{I}$ is the direct product of the quasi Jordan algebras $\mathfrak{I}$ and $R(\mathfrak{I})$. In the sequel, we will be mainly studying unital quasi Jordan algebras. The usual unitization process does not work for an arbitrary quasi Jordan algebra; however, one can attach a unit to any split quasi Jordan algebra and for any split quasi Jordan algebra $\mathfrak{I}$ with a (right) unit e, there exists a unique element $e_J \in J$, which is a unit of the quasi Jordan algebra $\mathfrak{I}$ as well as the unique (two-sided) unit of the Jordan algebra J (cf. [25]).

Quasi Jordan Banach Algebras. A real or complex quasi Jordan algebra $(\mathfrak{I}, \lhd)$ endowed with a norm $\|.\|$ is called a quasi Jordan normed algebra if $\|x \lhd y\| \leq \|x\|\|y\|$, for all $x, y \in \mathfrak{I}$. A quasi Jordan normed algebra is called a quasi Jordan Banach algebra if it is complete with respect to the metric induced by its norm. It is known from [1] that the Jordan part J of any unital split quasi Jordan Banach algebra $\mathfrak{I} = \mathfrak{I} \oplus \mathfrak{Z}(\mathfrak{I})$, with unit $e \in J$, is a norm closed subalgebra of $\mathfrak{I}$, and hence a unital Jordan Banach algebra [1, Proposition 2]. If $\mathfrak{I}$ is a quasi Jordan Banach algebra with a unit e of norm 1, then the quasi Jordan algebra $R(\mathfrak{I})$ is a quasi Jordan Banach algebra with unit R_e of norm 1; the split quasi Jordan algebra $\mathfrak{I}$ is a quasi Jordan Banach algebra with unit (e, R_e); $\varphi(\mathfrak{I}) = \{(\mathfrak{x}, \mathfrak{R}_\mathfrak{x})\ \ \mathfrak{x} \in \mathfrak{I}\}$ is a closed unital quasi Jordan normed subalgebra of $\mathfrak{I}$; and that any quasi Jordan Banach algebra $\mathfrak{I}$ with a norm 1 unit can be given an equivalent norm that makes the algebra $\mathfrak{I}$ isometrically isomorphic to a norm closed right ideal of a unital split quasi Jordan Banach algebra [1, propositions 3, 4, 5].

An element a in a quasi Jordan algebra $\mathfrak{I}$ is called invertible with respect to a unit $e \in \mathfrak{I}$ if there exists $b \in \mathfrak{I}$ such that $b \lhd a = e + e_\lhd(a)$ and $b \lhd a^2 = a + e_\lhd(a) + e_\lhd(a^2)$, where $e_\lhd(x) := e \lhd x - x$. As usual, the spectrum $\sigma_{(\mathfrak{I},e)}(x)$ of x in a complex quasi Jordan algebra $\mathfrak{I}$ with a fixed unit e is defined by $\sigma_{(\mathfrak{I},e)}(x) := \{\lambda \in \mathbb{C} : \lambda e - x$ is not invertible with respect to e$\}$ where $\mathbb{C}$ denotes the field of complex numbers. These definitions are consistent with the usual ones for Jordan algebras; however, in contrast to the classical theory of Jordan Banach algebras, the set of invertible elements in a unital quasi Jordan Banach algebra may not be open and that the spectrum of any element with respect to any fixed unit is nonempty but it may neither be bounded nor closed, hence not compact; moreover, if the spectrum of an element in a complex unital split quasi Jordan Banach algebra is unbounded then it coincides with the whole complex plane (cf. [1]).

2 Involution on a Quasi Jordan Algebra

Classically, an involution is a self-map g satisfying the identity $g^2(x) = x$. The identity maps and the complex conjugation are simple examples of (multiplicative) involutions. In functional analysis, the involutions frequently studied are often reverse multiplicative; sometimes also called anti-involution. Such an important example is the involution of a C^*-algebra; more specifically, the adjoint of a bounded linear operator on a Hilbert space. However, the involution of any C^*-algebra considered as a JB^*-algebra is multiplicative. More generally, if $*$ is a reverse multiplicative involution on an associative linear algebra A, then the same $*$ becomes a multiplicative involution on A considered as Jordan algebra. Indeed, any Jordan algebra is a quasi Jordan algebra and any involution on a Jordan algebra is multiplicative.

In [4], to study dialgebras from the functional analytical point of view, R. Felipe defined involutions of types *I*, *II* and *III* on a complex dialgebra as follows:

Definition 2.1 Let $(\mathcal{D}, \dashv, \vdash)$ be a complex dialgebra. A map $x \mapsto x^*$ of $\mathcal{D}$ onto itself is called an involution of type *I* if the following conditions are satisfied for all $x, y \in \mathcal{D}$ and $\alpha \in \mathbb{C}$:

(i). $(x + y)^* = x^* + y^*$;
(ii). $(\alpha x)^* = \overline{\alpha} x^*$;
(iii). $(x^*)^* = x$;
(iv). $(x \vdash y)^* = y^* \dashv x^*$.

Note that the above conditions (*iii*) and (*iv*) together give $(x \dashv y)^* = y^* \vdash x^*$, for all $x, y \in \mathcal{D}$. The map $x \mapsto x^*$ is said to be an involution of type *II* if it satisfies all the above conditions (*i*)–(*iii*) and the condition $(iv)'$ (in place of (*iv*)) stated as below:

$(iv)'$. $(x \vdash y)^* = y^* \vdash x^*$.

If the condition (*iv*) is replaced by the following condition:

$(iv)''$. $(x \dashv y)^* = y^* \dashv x^*$,

then the map "$*$" is called an involution of type *III*.

A complex dialgebra with an involution of type *I* (respectively, type *II* or type *III*) is called a $*$-dialgebra of type *I* (respectively, type *II* or type *III*).

Any $*$-dialgebra D of type *II* or *III* with a bar-unit is necessarily of type *I*; the converse is not true (cf. [4]). It is known that any complex dialgebra with a bar-unit and an involution of type *II* or *III* is necessarily an associative algebra (cf. [4]); hence, the induced plus quasi Jordan algebra must be a complex Jordan algebra with multiplicative involution. In the following discussion, we will see that any involution of type *I* on a complex dialgebra D is a multiplicative involution but may not be reverse multiplicative on the induced quasi Jordan algebra D^+ (see Definition 2.2 and Example 2.5). Moreover, if a unital quasi Jordan algebra $\mathfrak{I}$ has a reverse multiplicative involution then $\mathfrak{I}$ must be a Jordan algebra (see Remark 2.3); thus, the theory of unital quasi Jordan algebras with reverse multiplicative involution would coincide with the classical theory of Jordan algebras with involution.

In view of the above facts and to keep our study of involutions in the general setting of complex quasi Jordan algebras, we shall require the involutions to be multiplicative. We proceed with the following definition of an involution on a quasi Jordan algebra:

Definition 2.2 A self-map $* : x \mapsto x^*$ on a complex quasi Jordan algebra $\mathfrak{I}$ is called an involution on $\mathfrak{I}$ if it satisfies the following conditions:

(i). $(x^*)^* = x$;
(ii). $(x + y)^* = x^* + y^*$;
(iii). $(\alpha x)^* = \overline{\alpha} x^*$;
(iv). $(x \triangleleft y)^* = x^* \triangleleft y^*$

for all $x, y \in \mathfrak{I}$ and $\alpha \in \mathbb{C}$. In such a case,$\mathfrak{I}$ is called a quasi Jordan $*$-algebra. If, in addition, the algebra $\mathfrak{I}$ is normed satisfying the condition:

(v). $\|x^*\| = \|x\|$, for all $x \in \mathfrak{I}$,
then $\mathfrak{I}$ is called an involutive quasi Jordan normed algebra.

Remark 2.3 Suppose $\mathfrak{I}$ is a quasi Jordan algebra with unit e equipped with a map $* : \mathfrak{I} \to \mathfrak{I}$ satisfying the conditions $(i) - (iii)$ in the above definition 2.2 and $(x \triangleleft y)^* = y^* \triangleleft x^*$ for all $x, y \in \mathfrak{I}$. Then $x \triangleleft y = (x \triangleleft y) \triangleleft e = (e^* \triangleleft (x \triangleleft y)^*)^* = (e^* \triangleleft (y^* \triangleleft x^*))^* = (e^* \triangleleft (x^* \triangleleft y^*))^* = (e^* \triangleleft (y \triangleleft x)^*)^* = (y \triangleleft x) \triangleleft e = y \triangleleft x$ for all $x,y \in \mathfrak{I}$. Hence, $\mathfrak{I}$ is a Jordan algebra.

Example 2.4 Any involutive (complex) Jordan normed algebra is an involutive quasi Jordan normed algebra. In particular, all JB^*-algebras and so all C^* -algebras are involutive quasi Jordan Banach algebras.

Next example gives a class of quasi Jordan $*$-algebras with involution which may not be Jordan algebras.

Example 2.5 Let $(D, \dashv, \vdash)$ be a complex dialgebra with an involution $*$ of type I . Then $(x \vdash y)^* = y^* \dashv x^*$ and $(x \dashv y)^* = y^* \vdash x^*$. Hence,

$$\begin{aligned}(x \triangleleft y)^* &= \frac{1}{2}(x \dashv y + y \vdash x)^* \\ &= \frac{1}{2}((x \dashv y)^* + (y \vdash x)^*) = \frac{1}{2}(y^* \vdash x^* + x^* \dashv y^*) \\ &= \frac{1}{2}(x^* \dashv y^* + y^* \vdash x^*) = x^* \triangleleft y^*,\end{aligned}$$

for all $x, y \in \mathcal{D}$. Therefore, $*$ is an involution on the plus quasi Jordan algebra D^{+}; this involution may not be reverse multiplicative because the induced quasi Jordan product $\triangleleft$ in D^{+}, defined by $x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x)$ for all $x, y \in D$, generally is not commutative.

Remark 2.6 Suppose a dialgebra $(D, \dashv, \vdash)$ has an involution of type *II* which is also of type *III*. Then:

$$\begin{aligned}(x \triangleleft y)^{*} &= \frac{1}{2}(x \dashv y + y \vdash x)^{*} \\ &= \frac{1}{2}(y^{*} \dashv x^{*} + x^{*} \vdash y^{*}) \\ &= y^{*} \triangleleft x^{*}.\end{aligned}$$

Further, if the dialgebra $\mathcal{D}$ has a bar-unit e. Then e is a unit in $(\mathcal{D}, \triangleleft)$:

$$\begin{aligned}x \triangleleft e &= \frac{1}{2}(x \dashv e + e \vdash x) \\ &= \frac{1}{2}(x + x) = x.\end{aligned}$$

Hence, by Remark 2.3, $(\mathcal{D}, \triangleleft)$ is a Jordan algebra.

Lemma 2.7 *Let $\mathfrak{J}$ be a quasi Jordan $*$-algebra. Then $* : R_x \mapsto R_x^{*} := R_{x^{*}}$ is an involution on the quasi Jordan algebra $R(\mathfrak{J})$.*

Proof Clearly, for any $x, y \in \mathfrak{J}$ and $\lambda, \beta \in \mathbb{C}$:

$$\begin{aligned}\left(\lambda R_x + \beta R_y\right)^{*} &= R^{*}_{\lambda x + \beta y} = R_{(\lambda x + \beta y)^{*}} = R_{\overline{\lambda} x^{*} + \overline{\beta} y^{*}} \\ &= \overline{\lambda} R_{x^{*}} + \overline{\beta} R_{y^{*}} = \overline{\lambda} R_x^{*} + \overline{\beta} R_y^{*}, \\ \left(R_x^{*}\right)^{*} &= (R_{x^{*}})^{*} = R_{(x^{*})^{*}} = R_x\end{aligned}$$

and

$$\left(R_x \triangleleft R_y\right)^{*} = R^{*}_{x \triangleleft y} = R_{(x \triangleleft y)^{*}} = R_{x^{*} \triangleleft y^{*}} = R_{x^{*}} \triangleleft R_{y^{*}} = R_x^{*} \triangleleft R_y^{*}.$$

□

Theorem 2.8 *Let $\mathfrak{J}$ be an involutive quasi Jordan normed algebra with unit e of norm 1. Then $R(\mathfrak{J})$ is an involutive quasi Jordan normed algebra with unit R_e of norm 1.*

Proof By [1, Proposition 3], $R(\mathfrak{J})$ is a quasi Jordan normed algebra with bounded operator norm and unit R_e of norm 1. By the previous lemma, $R_x^{*} := R_{x^{*}}$ is an involution on the quasi Jordan algebra $R(\mathfrak{J})$. Moreover,

$$\begin{aligned}\|R_x^{*}\| &= \|R_{x^{*}}\| = \sup_{\substack{y \in \mathfrak{J} \\ \|y\| \neq 0}} \frac{\|y \triangleleft x^{*}\|}{\|y\|} = \sup_{\substack{y \in \mathfrak{J} \\ \|y^{*}\| \neq 0}} \frac{\|(y \triangleleft x^{*})^{*}\|}{\|y^{*}\|} \\ &= \sup_{\substack{y \in \mathfrak{J} \\ \|y^{*}\| \neq 0}} \frac{\|y^{*} \triangleleft x\|}{\|y^{*}\|} = \sup_{\substack{y \in \mathfrak{J} \\ \|y\| \neq 0}} \frac{\|y \triangleleft x\|}{\|y\|} = \|R_x\|.\end{aligned}$$

□

Theorem 2.9 *If $\mathfrak{J}$ is an involutive quasi Jordan normed algebra then so is $\mathfrak{J} \oplus \mathfrak{R}(\mathfrak{J})$.*

Proof Recall from [24] that if $\mathfrak{J}$ a quasi Jordan algebra then $\mathfrak{J} \oplus \mathfrak{R}(\mathfrak{J})$ together with the product $\left(x + R_y\right) \triangleleft (z + R_w) = x \triangleleft w + R_{y \triangleleft w}$ is a quasi Jordan algebra. We define $*$ on $\mathfrak{J} \oplus \mathfrak{R}(\mathfrak{J})$ by $\left(x + R_y\right)^{*} = x^{*} + R_y^{*} = x^{*} + R_{y^{*}}$. One can easily verify that $*$ is conjugate linear and

$$\begin{aligned}\left((x+R_y)\lhd(z+R_w)\right)^* &= \left(x\lhd w+R_{y\lhd w}\right)^*\\ &= x^*\lhd w^*+R^*_{y\lhd w}\\ &= x^*\lhd w^*+R_{y^*\lhd w^*}\\ &= \left(x^*+R_{y^*}\right)\lhd(z^*+R_{w^*})\\ &= \left(x+R_y\right)^*\lhd(z+R_w)^*.\end{aligned}$$

So, $*$ is an involution on $\mathfrak{I}\oplus\mathfrak{R}(\mathfrak{I})$. Moreover, if $\mathfrak{I}$ is an involutive quasi Jordan normed (Banach) algebra then $\mathfrak{I}\oplus\mathfrak{R}(\mathfrak{I})$ with $\|x+R_y\|=\|x\|+\|R_y\|$ is a quasi Jordan normed (Banach) algebra by [1, Proposition 4]. Since $\mathfrak{I}$ is involutive, so is $R(\mathfrak{I})$ by Theorem 2.8. Hence,

$$\|(x+R_y)^*\|=\left\|x^*+R^*_y\right\|=\|x^*\|+\left\|R^*_y\right\|=\|x\|+\|R_y\|=\|x+R_y\|.$$

□

Let $\mathfrak{I}$ be a split quasi Jordan $*$-algebra. Let $\overline{\mathfrak{I}}=\mathfrak{I}\times\mathbb{C}$ be the unitization of $\mathfrak{I}$ (cf. [[25], Lemma 38]) with the quasi Jordan product defined by:

$$\begin{aligned}(x,\alpha)\lhd(y,\beta) &= (x_J+x_Z,\alpha)\lhd(y_J+y_Z,\beta)\\ &= (x_J\lhd y_J+\beta x_J+\beta x_Z+\alpha y_J,\alpha\beta).\end{aligned}$$

We define $*$ on $\overline{\mathfrak{I}}$ by $(x,\alpha)^*=(x^*,\overline{\alpha})$. The conjugate linearity of $*$ on $\overline{\mathfrak{I}}$ is clear. Further,

$$\begin{aligned}((x,\alpha)\lhd(y,\beta))^* &= (x_J\lhd y_J+\beta x_J+\beta x_Z+\alpha y_J,\alpha\beta)^*\\ &= \left(x^*_J\lhd y^*_J+\overline{\beta}x^*_J+\overline{\beta}x^*_Z+\overline{\alpha}y^*_J,\overline{\alpha}\overline{\beta}\right)\\ &= (x^*,\overline{\alpha})\lhd\left(y^*,\overline{\beta}\right)\\ &= (x,\alpha)^*\lhd(y,\beta)^*,\end{aligned}$$

for all scalars α,β and $x,y\in\mathfrak{I}$. Hence, $*$ is an involution on $\overline{\mathfrak{I}}$. Further, if $\|(x,\alpha)\|=\|x\|+|\alpha|$ is a norm on on the algebra $\overline{\mathfrak{I}}$, then:

$$\begin{aligned}\|(x,\alpha)^*\| &= \|(x^*,\overline{\alpha})\|=\|x^*\|+|\overline{\alpha}|\\ &= \|x\|+|\alpha|=\|(x,\alpha)\|.\end{aligned}$$

Thus, $\overline{\mathfrak{I}}$ is an involutive quasi Jordan normed algebra.

From the above discussion, it is clear that if $\mathfrak{I}$ is a quasi Jordan algebra satisfying the conditions $(i)-(iii)$ of Definition 2.2 then $\mathfrak{I}$ can be imbedded in a unital split quasi Jordan algebra that also satisfies the conditions $(i)-(iii)$. If $*$ on $\mathfrak{I}$ is reverse multiplicative then, by the argument given in Remark 2.3, $\mathfrak{I}$ must be a Jordan algebra.

We close this section with the following result:

Theorem 2.10 *Let $\mathfrak{I}$ be a quasi Jordan normed $*$-algebra.*

(1) If $\|x^*\lhd x\|=\|x\|^2$, for all $x\in\mathfrak{I}$, then $\mathfrak{I}$ is a Jordan algebra.
(2) If $\|\{xx^*x\}\|=\|x\|^3$, for all $x\in\mathfrak{I}$, then $\mathfrak{I}$ is a Jordan algebra. Here,$\{abc\}:=a\lhd(b\lhd c)-(a\lhd c)\lhd b+(a\lhd b)\lhd c$.

Proof

(1) Let $\|x^*\lhd x\|=\|x\|^2$, for all $x\in\mathfrak{I}$. Then, for any $z\in\mathcal{Z}(\mathfrak{I})$, $\|z\|^2=\|z^*\lhd z\|=\|0\|=0$. That is, $\mathcal{Z}(\mathfrak{I})=\{0\}$. Recall that $\mathfrak{I}^{ann}\subseteq\mathfrak{Z}(\mathfrak{I})$. Hence, $\mathfrak{I}^{ann}=\{\}$. Thus, $\mathfrak{I}$ is a Jordan algebra.
(2) Let $\|\{xx^*x\}\|=\|x\|^3$, for all $x\in\mathfrak{I}$. We observe that $z^*\in\mathcal{Z}(\mathfrak{I})$ whenever $z\in\mathcal{Z}(\mathfrak{I})$ because $x\lhd z^*=(x^*\lhd z)^*=0^*=0$, for all $x\in\mathfrak{I}$. Hence, for any $z\in\mathcal{Z}(\mathfrak{I})$,

$$\begin{aligned}\|z\|^3 &= \|\{zz^*z\}\|=\|z\lhd(z^*\lhd z)-(z\lhd z)\lhd z^*+(z\lhd z)\lhd z^*\|\\ &= \|z\lhd 0-0\lhd z^*+0\|=\|0\|=0.\end{aligned}$$

Thus, $\mathcal{Z}(\mathfrak{I}) = \{0\}$.

$\square$

3 Self-adjoint Elements

Here, we begin with the following usual definition:

Definition 3.1 An element x in a quasi Jordan $*$-algebra $\mathfrak{I}$ is said to be self-adjoint if $x^* = x$. The set of all self-adjoint elements in $\mathfrak{I}$ is denoted by $\mathfrak{I}_{sa}$.

Remark 3.2 In any Jordan $*$-algebra, an element is self-adjoint if and only if its spectrum is contained in $\mathbb{R}$. This is not true in the case of quasi Jordan algebras. By [1, Proposition 12], the spectrum of any zero element in a unital quasi Jordan algebra is $\{0\}$. However, the following theorem tells also that a quasi Jordan $*$-algebra may have nonself-adjoint zero elements.

Theorem 3.3 *Let $\mathfrak{I}$ be a quasi Jordan $*$-algebra. Then Then both the sets $U(\mathfrak{I})$ and $\mathcal{Z}(\mathfrak{I})$ are self-adjoint (that is, closed under $*$). If $\mathfrak{I}$ is not a Jordan algebra then $\mathfrak{I}$ must have at least one nonself-adjoint zero element. Moreover, if $\mathfrak{I}$ is unital then it may have nonself-adjoint units.*

Proof If $e \in U(\mathfrak{I})$ then $x \lhd e^* = (x^* \lhd e)^* = (x^*)^* = x$ for all $x \in \mathfrak{I}$; so that $e^* \in U(\mathfrak{I})$.

Next, if $z \in \mathcal{Z}(\mathfrak{I})$ then $z^* \in \mathcal{Z}(\mathfrak{I})$ as seen in the proof of Theorem 2.10.

Now, suppose every zero element in $\mathfrak{I}$ is self-adjoint. Then, for an arbitrary $z \in \mathcal{Z}(\mathfrak{I})$, we have $z = z^*$ and $iz \in \mathcal{Z}(\mathfrak{I})$; hence, $iz = (iz)^* = -iz^* = -iz$ and so $z = 0$. Therefore, $\mathcal{Z}(\mathfrak{I}) = \{0\}$; which in turn implies that $\mathfrak{I}$ is a Jordan algebra.

Further, if e is a unit in $\mathfrak{I}$ is then $i(e - e^*)$ is a self-adjoint zero element and $\frac{1}{2}(e + e^*)$ is a self-adjoint unit because e^* is a unit in $\mathfrak{I}$ (as seen above).

Finally, If the quasi Jordan $*$-algebra $\mathfrak{I}$ is unital but not a Jordan algebra then from the above discussion it is clear that $\mathfrak{I}$ has a nonself-adjoint zero element z and a self-adjoint unit e; hence, $(z + e) \in \{x + e : x \in Z(\mathfrak{I})\} = U(\mathfrak{I})$ (see Section 1) but $(z + e)^* = z^* + e^* = z^* + e \neq z + e$ since $z^* \neq z$. $\square$

Theorem 3.4 *Let $\mathfrak{I} = \mathfrak{I} \oplus \mathcal{Z}(\mathfrak{I})$ be a unital split quasi Jordan algebra with involution $*$. Then J and $\mathcal{Z}(\mathfrak{I})$ are self-adjoint.*

Proof The zero part $\mathcal{Z}(\mathfrak{I})$ is self-adjoint by Theorem 3.3.

For any $x \in J$, we have $x^* = (x^*)_J + (x^*)_z \Longrightarrow x = x^{**} = \left((x^*)_J\right)^* + \left((x^*)_z\right)^*$. By Theorem 3.3, $\left((x^*)_z\right)^* \in \mathcal{Z}(\mathfrak{I})$, and so by the uniqueness of representations in $\mathfrak{I}$, $\left((x^*)_z\right)^* = 0$. Hence, $x^* \in J$. Thus, J is closed under the involution $*$. $\square$

Remark 3.5 Let $\mathfrak{I}$ be a quasi Jordan $*$ -algebra. Then:

(1) Since $\mathcal{Z}(\mathfrak{I})$ is self-adjoint, we have $z \lhd z^* = 0 = z^* \lhd z$ for all $z \in \mathcal{Z}(\mathfrak{I})$. However, $e \lhd e^* = e \neq e^* = e^* \lhd e$ if e is not self-adjoint.
(2) If $\mathfrak{I}$ is a split quasi Jordan algebra with unit e then e_J being the unique unit in the Jordan algebra J (that is, the Jordan part of $\mathfrak{I}$) is self-adjoint since $U(J)$ is self-adjoint by Theorem 3.3.
(3) If $\mathfrak{I}$ is unital, then there exist a self-adjoint unit $\frac{1}{2}(e + e^*)$ and a self-adjoint zero $i(e - e^*)$; clearly, the zero element $e - e^*$ is not self-adjoint.
(4) If $\mathfrak{I}$ is an involutive quasi Jordan normed algebra with $\|x^* \lhd x\| = \|x\|^2$ for all $x \in \mathfrak{I}$, then $\|x\| = \sup_{\|y\| \leq 1} \|x \lhd y\| = \sup_{\|y\| \leq 1} \|y \lhd x\|$. For this, let y be such that $\|y\| \leq 1$. Then we have $\|x \lhd y\| \leq \|x\| \|y\| \leq \|x\|$ and also $\|y \lhd x\| \leq \|x\|$. To prove the inequalities $\|x\| \leq \sup_{\|y\| \leq 1} \|x \lhd y\|$ and $\|x\| \leq \sup_{\|y\| \leq 1} \|y \lhd x\|$, we consider the case when $\|x\| = 1$. In this case, $\|x^*\| = 1$; so that $\sup_{\|y\| \leq 1} \|x \lhd y\| \geq \|x \lhd x^*\| = \|x^*\|^2 = \|x\|^2 = 1$ and $\sup_{\|y\| \leq 1} \|y \lhd x\| \geq \|x^* \lhd x\| = \|x\|^2 = 1$. Now, if $\|x\| = 0$, the equality is clear. Next, if $\|x\| \neq 0$, then we put $z = \|x\|^{-1} x$ and get that

$$\begin{aligned}1 &= \|x\|^{-1}\|x\| = \|z\| \\ &= \sup_{\|y\|\leq 1}\|z \lhd y\| \\ &= \sup_{\|y\|\leq 1}\left\|\left(\|x\|^{-1}x\right)\lhd y\right\| \\ &= \|x\|^{-1}\sup_{\|y\|\leq 1}\|x \lhd y\|.\end{aligned}$$

Hence, $\|x\| = \sup_{\|y\|\leq 1}\|x \lhd y\|$. Similarly, $\|x\| = \sup_{\|y\|\leq 1}\|y \lhd x\|$.

(5) If $\mathfrak{I}$ is an involutive Jordan normed algebra with unit e satisfying $\|x^* \lhd x\| = \|x\|^2$ for all $x \in \mathfrak{I}$, then $\|e\|^2 = \|e^* \lhd e\| = \|e^*\| = \|e\|$, so that $\|e\| = 1$.

Theorem 3.6 *Let $\mathfrak{I}$ be a quasi Jordan $*$-algebra with a unit e and $x \in \mathfrak{I}$. Then*:

(1) $x + x^*$ and $i(x - x^*)$ are self-adjoint.
(2) x has a unique representation $x = u + iv$, where u and v are self-adjoint elements of $\mathfrak{I}$.
(3) If x is invertible w.r.t. the unit e with inverse $y \in \mathfrak{I}$ then x^* is invertible w.r.t. the unit e^* with inverse y^*. Hence, the set of invertible elements w.r.t. the unit e in $\mathfrak{I}$ is self-adjoint if e is self-adjoint.
(4) $\lambda \in \sigma_{(\mathfrak{I},e)}(x)$ if and only if $\overline{\lambda} \in \sigma_{(\mathfrak{I},e^*)}(x^*)$. Hence,$\lambda \in \sigma_{(\mathfrak{I},e)}(x)$ if and only if $\overline{\lambda} \in \sigma_{(\mathfrak{I},e)}(x^*)$ in the case when e is self-adjoint.

Proof

(1) Clearly, $(x + x^*)^* = x^* + (x^*)^* = x^* + x$ and $(i(x - x^*))^* = -i((x - x^*)^*) = -i(x^* - x) = i(x - x^*)$.
(2) Let $u = \frac{1}{2}(x + x^*)$ and $v = \frac{i}{2}(x^* - x)$. Then $x = u + iv$, where both u and v are self-adjoint. Suppose $x = u' + iv'$, where u' and v' are self-adjoint elements. Put $w = v' - v$. Then both w and iw are self-adjoint, so that $iw = (iw)^* = -iw^* = -iw$. Hence $w = 0$, and the uniqueness follows.
(3) Clearly,

$$y^* \lhd x^* = (y \lhd x)^* = (e + e \lhd x - x)^* = e^* + e^* \lhd x^* - x^* = e^* + e^*_{\lhd}(x^*),$$

and

$$\begin{aligned}y^* \lhd (x^*)^2 &= \left(y \lhd x^2\right)^* \\ &= \left(x + (e \lhd x - x) + \left(e \lhd x^2 - x^2\right)\right)^* \\ &= x^* + (e^* \lhd x^* - x^*) + \left(e^* \lhd (x^*)^2 - (x^*)^2\right) \\ &= x^* + e^*_{\lhd}(x^*) + e^*_{\lhd}\left((x^*)^2\right).\end{aligned}$$

(4) Let $\lambda \in \sigma_{(\mathfrak{I},e)}(x)$. Then $\lambda e - x$ is not invertible w.r.t. e. By the above part (3), $(\lambda e - x)^* = \overline{\lambda}e^* - x^*$ is also not invertible w.r.t. e^*. Hence, $\overline{\lambda} \in \sigma_{(\mathfrak{I},e^*)}(x^*)$.

□

We recall that $\mathfrak{I}_{\mathfrak{sa}} = \{\mathfrak{x} \in \mathfrak{I} \ \ \mathfrak{x}^* = \mathfrak{x}\}$.

Theorem 3.7 *If $\mathfrak{I}$ is an involutive quasi Jordan Banach algebra, then $\mathfrak{I}_{\mathfrak{sa}}$ is a real quasi Jordan Banach subalgebra of $\mathfrak{I}$.*

Proof Let $x, y \in \mathfrak{I}_{\mathfrak{sa}}$. Then $x^* = x$ and $y^* = y$ and so

$$\begin{aligned}(ax)^* &= \overline{a}x^* = ax \\ (x + y)^* &= x^* + y^* = x + y \\ (x \lhd y)^* &= x^* \lhd y^* = x \lhd y,\end{aligned}$$

for any $a \in \mathbb{R}$. Hence, $ax, x + y, x \lhd y \in \mathfrak{I}_{\mathfrak{sa}}$ whenever $x, y \in \mathfrak{I}_{\mathfrak{sa}}$ and $a \in \mathbb{R}$.

Next, we show that $\mathfrak{I}_{\mathfrak{sa}}$ is norm closed. Let (x_n) be a sequence in $\mathfrak{I}_{\mathfrak{sa}}$ that converges to $x \in \mathfrak{I}$. Since the involution $*$ on $\mathfrak{I}$ is an isometry, it is continuous. Hence, $x_n^* \to x^*$ as $n \to \infty$. But $x_n^* = x_n$ for all n, so $x_n \to x^*$. Therefore, by the uniqueness of the limit $x = x^*$. Thus, $x \in \mathfrak{I}_{\mathfrak{sa}}$. □

4 Unitary Elements

In this section, we introduce the notions of e-unitary elements and unitary elements in the setting of unital quasi Jordan algebras with an involution. In case of Jordan algebras, the two notions (e-unitary elements and unitary elements) coincide with the usual notion of unitary elements. We investigate various properties of these unitaries and intend in the sequel to obtain some appropriate analogue of the Russo-Dye theorem for unital involutive split quasi Jordan Banach algebras.

Definition 4.1 An element u in a unital quasi Jordan algebra with involution $*$ is said to be e-unitary if it is invertible with respect to e with inverse u^*. An element u is said to be unitary if it is e-unitary for all units $e \in \mathfrak{I}$. We will denote the collection of e-unitaries in $\mathfrak{I}$ by $\mathcal{U}_e(\mathfrak{I})$ and $\mathcal{U}(\mathfrak{I})$ for unitary elements in $\mathfrak{I}$. Thus $\mathcal{U}(\mathfrak{I})$ is the intersection of $\mathcal{U}_e(\mathfrak{I})$ for all units e in $\mathfrak{I}$.

We note that if u is unitary then u^* is also unitary:since u is a unitary then there is a unit e in $\mathfrak{I}$ such that $u^* \lhd u = e + e_{\lhd}(u) = e + e \lhd u - u$ and hence:

$$\begin{aligned}(u^*)^* \lhd u^* =&(u^* \lhd u)^*\\ =&(e + e \lhd u - u)^*\\ =&e^* + e^* \lhd u^* - u^*\\ =&e^* + e^*_{\lhd}(u^*).\end{aligned}$$

where for any unit e, e^* is also a unit (see Theorem 3.3). In a similar manner one can show that:

$$(u^*)^* \lhd (u^*)^2 = u^* + e_{\lhd}(u^*) + e_{\lhd}\left(u^{*^2}\right).$$

Any self-adjoint unit e in a quasi Jordan $*$-algebra $\mathfrak{I}$ is an e -unitary because $e_{\lhd}(e) = e \lhd e - e = 0$ and $e_{\lhd}e^2 = e \lhd e^2 - e^2 = e \lhd (e \lhd e) - e = e \lhd e - e = e - e = 0$ so that $e^* \lhd e = e^* = e = e + 0 = e + e_{\lhd}(e)$ and $e^* \lhd e^2 = e^* \lhd (e \lhd e) = e^* \lhd e = e^* = e = e + 0 + 0 = e + e_{\lhd}(e) + e_{\lhd}(e^2)$. Moreover, such a unit e is unitary if and only if $\mathcal{Z}(\mathfrak{I}) = \{0\}$ because for all $z \in \mathcal{Z}(\mathfrak{I})$, $e + z$ is a unit in $\mathfrak{I}$ and so $e + e \lhd e - e = e^* \lhd e = (e+z) + (e+z) \lhd e - e$ gives $0 = 2z$; which means $\mathcal{Z}(\mathfrak{I}) = \{0\}$ and hence $\mathfrak{I}$ is a Jordan algebra.

Theorem 4.2 *Let u be an e-unitary element in a unital split quasi Jordan algebra $\mathfrak{I} = \mathfrak{J} \oplus \mathcal{Z}(\mathfrak{I})$ with involution for some unit $e \in \mathfrak{I}$. Then $u_{_J}$ is a unitary element in the Jordan part J.*

Proof Clearly, we have $u^* = u^*_{_J} + u^*_{_Z}$. By the part 2 of Remark 3.5 the unit $e_{_J}$ of the Jordan part is self-adjoint. By Theorem 3.4, the Jordan part J) is self-adjoint. So that $u^*_{_J} \in J$. Hence, by the uniqueness of the representation $u^* = (u^*)_{_J} + (u^*)_{_Z}$ in $\mathfrak{I} = \mathfrak{J} \oplus \mathcal{Z}(\mathfrak{I})$, we get $u^*_{_J} = (u^*)_{_J}$. Thus, by [1, Proposition 14], $u^*_{_J}$ is the inverse of $u_{_J}$ in J; which means that $u_{_J}$ is a unitary in the Jordan part. □

Theorem 4.3 *Let $\mathfrak{I}$ be a unital quasi Jordan algebra with involution. Then $\mathcal{U}(\mathfrak{I})$ is not empty.*

Proof Let e be a self-adjoint unit and z a self-adjoint zero element in $\mathfrak{I}$. Then $u = -(e+z)$ is a unitary: for this, let $e_\circ$ be any unit in $\mathfrak{I}$. Then:

$$\begin{aligned}u^* \lhd u =& e + z = e_\circ + e_\circ \lhd u - u,\\ u^* \lhd u^2 =& -(e+z) = e_\circ + e_\circ \lhd u - u + e_\circ \lhd u^2 - u^2.\end{aligned}$$

□

Theorem 4.4 *Let $\mathfrak{I}$ be a unital quasi Jordan algebra with involution, and $u \in \mathcal{U}_e(\mathfrak{I})$ for some unit $e \in \mathfrak{I}$. Then, $u \in \mathcal{U}(\mathfrak{I})$ if and only if*

$$z \lhd u = -z \text{ for all } z \in \mathcal{Z}(\mathfrak{I}).$$

Proof Let $u \in \mathcal{U}(\mathfrak{I})$ and $z \in \mathcal{Z}(\mathfrak{I})$. Since $e + z$ is a unit, we have:

$$u^* \lhd u = e + e \lhd u - u,$$

as well as

$$u^* \lhd u = (e + z) + (e + z) \lhd u - u.$$

Hence:

$$z + z \lhd u = 0.$$

Conversely, since $u \in \mathcal{U}_e(\mathfrak{I})$, we have:

$$u^* \lhd u = e + e \lhd u - u$$

and

$$z \lhd u = -z \text{ for all } z \in \mathcal{Z}(\mathfrak{I}).$$

For any fixed unit $e_\circ$ in $\mathfrak{I}$, let $z := e_\circ - e$. Then:

$$\begin{aligned} u^* \lhd u =& e + e \lhd u - u \\ =& e + z - z + e \lhd u - u \\ =& e + z - (-z \lhd u) + e \lhd u - u \\ =& e_\circ + e_\circ \lhd u - u. \end{aligned}$$

$\square$

Theorem 4.5 *Let $\mathfrak{I} = \mathfrak{J} \oplus \mathcal{Z}(\mathfrak{I})$ be a unital split quasi Jordan algebra with involution. If $u = u_J + u_Z$ is a unitary w.r.t. the unit of the Jordan part $e \in J$, then so is $\alpha u_J + \beta u_Z$, for all $\beta \in \mathbb{C}$ and α a square root of 1 that satisfies $\alpha\overline{\beta} = \beta$.*

Proof Let $\mathfrak{I}, \mathfrak{u}, \alpha, \beta$ be as above. As u^* is the inverse of u with respect to e, we have:

$$\begin{aligned} & u_J^* \lhd u_J = e, u_Z^* \lhd u_J = -u_Z = e_\lhd(u), \\ & u_J^* \lhd u_J^2 = u_J, u_Z^* \lhd u_J = -u_Z \lhd u_J = e_\lhd\left(u^2\right), \\ & \quad (\alpha u_J + \beta u_Z)^* \lhd (\alpha u_J + \beta u_Z) = |\alpha|^2 \left(u_J^* \lhd u_J\right) + \overline{\beta}\alpha u_Z^* \lhd u_J \\ & = e - \beta u_Z = e + e_\lhd(\alpha u_J + \beta u_Z) \end{aligned}$$

and

$$\begin{aligned} & (\alpha u_J + \beta u_Z)^* \lhd (\alpha u_J + \beta u_Z)^2 = \alpha^2 (\alpha u_J + \beta u_Z)^* \lhd u_J^2 \\ & \quad = \alpha^2 |\alpha| u_J - \alpha^2 \overline{\beta} u_Z \lhd u_J = \alpha u_J - \alpha\beta u_Z \lhd u_J \\ & \quad = (\alpha u_J + \beta u_Z) + e_\lhd(\alpha u_J + \beta u_Z) + e_\lhd\left((\alpha u_J + \beta u_Z)^2\right). \end{aligned}$$

$\square$

5 A Russo-Dye Type Theorem

In this section, we begin a study of convex combinations of unitaries. We present some results on such combinations including Theorem 5.3, which may be considered as a Russo-Dye type theorem for unital involutive split quasi Jordan Banach algebras, which is consistent with the Russo-Dye Theorem for JB^*-algebras (cf. [18, 28]).

Theorem 5.1 *Let $\mathfrak{I}$ be a unital quasi Jordan algebra with involution and let e be a self-adjoint unit in $\mathfrak{I}$. Then $\mathcal{Z}(\mathfrak{I}) \subseteq co(\mathcal{U}_e(\mathfrak{I}))$.*

Proof Let z be any is self-adjoint element in $\mathcal{Z}(\mathfrak{I})$. We show that $e + iz$ and $z - e$ are e-unitaries, as follows: Clearly,

$$e_{\triangleleft}(e+iz) = e \triangleleft (e+iz) - (e+iz) = e \triangleleft e + ie \triangleleft z - e - iz = e + 0 - e - iz = iz$$

and $(e+iz)^2 = (e+iz) \triangleleft (e+iz) = (e+iz) \triangleleft e + (e+iz) \triangleleft iz = (e+iz) + 0 = e + iz$ so that $e_{\triangleleft}(e+iz)^2 = e \triangleleft (e+iz)^2 - (e+iz)^2 = e \triangleleft (e+iz) - (e+iz) = e \triangleleft e + e \triangleleft z - e - iz = e + 0 - e - iz = -iz$. Hence, $(e+iz)^* \triangleleft (e+iz) = (e^* + iz^*) \triangleleft (e+iz) = (e-iz) \triangleleft (e+iz) = (e-iz) \triangleleft e + 0 = e - iz = e + e_{\triangleleft}$ $(e+iz)$ and $(e+iz)^* \triangleleft (e+iz)^2 = (e^* - iz^*) \triangleleft (e+iz) = (e-iz) \triangleleft (e+iz) = (e-iz) \triangleleft e + (e-iz) \triangleleft iz = e - iz = e + iz - iz - iz = (e+iz) + e_{\triangleleft}(e+iz) + e_{\triangleleft}(e+iz)^2$.

Similarly, $(-e+z)^* \triangleleft (-e+z) = (-e+z) \triangleleft (-e) = e - z = e + e_{\triangleleft}(-e+z)$ and $(-e+z)^* \triangleleft (-e+z)^2 = -e + z = (-e+z) + e_{\triangleleft}(-e+z) + e_{\triangleleft}\Big((-e+z)^2\Big)$.

Now, let z be any element in $\mathcal{Z}(\mathfrak{I})$. We put $z_1 = i(z^* - z)$ and $z_2 = z + z^*$. Then both z_1, z_2 are self-adjoint. Hence, from the above discussion, $e + iz_1$ and $-e + z_2$ are e-unitaries. Thus, $z = \frac{1}{2}((e+iz_1) + (-e+z_2)) \in co(\mathcal{U}_e(\mathfrak{I}))$. □

Theorem 5.2 (*Convex hull of e-unitaries*) *Let $\mathfrak{I} = \mathfrak{J} \oplus \mathcal{Z}(\mathfrak{I})$ be a unital involutive split quasi Jordan Banach algebra with unit $e \in J$. Then*

$$\overline{co}(\mathcal{U}_e(\mathfrak{I})) = \overline{co}(\mathcal{U}_e(J)) \oplus \mathcal{Z}(\mathfrak{I}),$$

where $\overline{co}(\mathcal{U}_e(J))$ coincides with $\overline{co}(\mathcal{U}(J))$, the closed convex hull of all unitary elements in the Jordan algebra J w.r.t. the restriction of original involution "$*$".

Proof Of course, the Jordan part J is self-adjoint. Then, under the hypothesis, J is a unital involutive Jordan Banach algebra by [1, Proposition 2]. Note that every unitary element in the involutive Jordan algebra J is also an e-unitary in $\mathfrak{I}$: for if $u \in J$ unitary, then u^* is the inverse of u in J and that $e_{\triangleleft}(u) = 0 = e_{\triangleleft}(u^2)$, hence u^* is an inverse of u in $\mathfrak{I}$. Then $\overline{co}(\mathcal{U}_e(J)) \subseteq \overline{co}(\mathcal{U}_e(\mathfrak{I}))$. By Theorem , $\mathcal{Z}(\mathfrak{I}) \subseteq \overline{co}(\mathcal{U}_e(\mathfrak{I}))$. Hence, $co(\mathcal{U}_e(J)) \cup \mathcal{Z}(\mathfrak{I}) \subseteq \overline{co}(\mathcal{U}_e(\mathfrak{I}))$. Now, let $x \in \overline{co}(\mathcal{U}_e(J)) and z \in \mathcal{Z}(\mathfrak{I})$. Then, for any $\epsilon > 0$, there exists x_ϵ which is a convex combination of unitary elements in J such that $\|x - x_\epsilon\| < \epsilon$. We will first show that $x_\epsilon + z \in \overline{co}(\mathcal{U}_e(\mathfrak{I}))$ and then conclude that $x + z \in \overline{co}(\mathcal{U}_e(\mathfrak{I}))$, as follows:

For any $n \in \mathbb{N}$, $nz \in \mathcal{Z}(\mathfrak{I}) \subset co(\mathcal{U}_e(\mathfrak{I}))$ and $x_\epsilon \in co(\mathcal{U}_e(\mathfrak{I}))$, hence $\frac{n-1}{n}x_\epsilon + z = \frac{n-1}{n}x_\epsilon + \frac{1}{n}(nz) \in co(\mathcal{U}_e(\mathfrak{I}))$, but $\frac{n-1}{n}x_\epsilon + z \to x_\epsilon + z$ as $n \to \infty$; this means $x_\epsilon + z \in \overline{co}(\mathcal{U}_e(\mathfrak{I}))$. As ϵ was arbitrary, $x + z \in \overline{co}(\mathcal{U}_e(\mathfrak{I}))$. Hence, $\overline{co}(\mathcal{U}_e(J)) + \mathcal{Z}(\mathfrak{I}) \subseteq \overline{co}(\mathcal{U}_e(\mathfrak{I}))$.

Conversely, let $x \in co(\mathcal{U}_e(\mathfrak{I}))$, Then $x = x_J + x_Z$ is a convex combination of e-unitary elements of $\mathfrak{I}$, say $x_J + x_Z = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$. For each $i = 1, .., n$, we have $u_i = (u_i)_J + (u_i)_Z$ with $(u_i)_J \in J$ and $(u_i)_Z \in \mathcal{Z}(\mathfrak{I})$. By Theorem 4.2, each $(u_i)_J$ is a unitary in J, and so by the uniqueness of representations $u_i = (u_i)_J + (u_i)_Z$ the direct sum $J \oplus \mathcal{Z}(\mathfrak{I})$, $x_J \in co(\mathcal{U}_e(J))$. This implies $co(\mathcal{U}_e(\mathfrak{I})) \subseteq \overline{co}(\mathcal{U}_e(J)) \oplus \mathcal{Z}(\mathfrak{I})$. However, $\overline{co}(\mathcal{U}_e(J)) \oplus \mathcal{Z}(\mathfrak{I})$ is norm closed. Therefore, $\overline{co}(\mathcal{U}_e(\mathfrak{I})) \subseteq \overline{co}(\mathcal{U}_e(J)) \oplus \mathcal{Z}(\mathfrak{I})$. □

Now, we give the following extension of the Russo-Dye theorem for unital involutive split quasi Jordan algebras:

Theorem 5.3 (*Russo-Dye type*) *Let $\mathfrak{I} = \mathfrak{J} \oplus \mathcal{Z}(\mathfrak{I})$ be an involutive unital split quasi Jordan Banach algebra,$e \in J$ a norm one unit . If the Jordan part J of $\mathfrak{I}$ is a JB^*-algebra, then $\overline{co}(\mathcal{U}_e(\mathfrak{I})) = B_1 \oplus \mathcal{Z}(\mathfrak{I})$, where B_1 is the closed unit ball of the Jordan part J. Moreover, $\overline{co}(\mathcal{U}(\mathfrak{I})) \subseteq B_1 \oplus \mathcal{Z}(\mathfrak{I})$*

Proof If the Jordan part J of $\mathfrak{I}$ is a JB^*-algebra, then by the Russo-Dye theorem for JB^*-algebras (cf. [18, 28]), we get $\overline{co}(\mathcal{U}_e(J)) = B_1$ (the closed unit ball of the Jordan part J). Hence, by Theorem 5.2, $\overline{co}(\mathcal{U}_e(\mathfrak{I})) = B_1 \oplus \mathcal{Z}(\mathfrak{I})$.

Next, we show that $B_1 \oplus Z(\mathfrak{I}) = \mathcal{B} \oplus \mathfrak{Z}(\mathfrak{I})$, where $\mathcal{B}_1$ denotes the closed unit balls of $\mathfrak{I}$: Clearly, $B_1 \subseteq \mathcal{B}_1$; so that $B_1 \oplus Z(\mathfrak{I}) \subseteq \mathcal{B}_1 \oplus Z(\mathfrak{I})$. Now, let $y \in \mathcal{B}_1 \oplus Z(\mathfrak{I})$. Then $y = x + y_Z$ with $x \in \mathcal{B}_1$ and $y_Z \in Z(\mathfrak{I})$. So that $x = x_J + x_Z$ with $x_J \in J$ and $x_Z \in Z(\mathfrak{I})$. Moreover, $\|x_J\| = \|e \triangleleft x\| \leq \|e\|\|x\| \leq 1$ (recall that e is the norm 1 unit in the Jordan algebra J). Hence, $y = x_J + x_Z + y_Z \in B_1 \oplus Z(\mathfrak{I})$ since $x_Z + y_Z \in Z(\mathfrak{I})$. This means $\mathcal{B}_1 \oplus Z(\mathfrak{I}) \subseteq B_1 \oplus Z(\mathfrak{I})$. Therefore, $B_1 \oplus Z(\mathfrak{I}) = \mathcal{B}_1 \oplus Z(\mathfrak{I})$.

Thus, $\overline{co}(\mathcal{U}_e(\mathfrak{I})) = \mathcal{B}_1 \oplus \mathcal{Z}(\mathfrak{I})$. However,

$$\mathcal{U}(\mathfrak{I}) = \cap\{\mathcal{U}_v(\mathfrak{I}) : v \text{is a unit in } \mathfrak{I}\}.$$

Therefore, the required inclusion follows. □

Remark 5.4 For a split quasi Jordan Banach algebra $\mathfrak{J}$, the two balls B_1 and $\mathcal{B}_1$ may not coincide because $\|z\|^{-1}z \in \mathcal{B}_1$ but $\|z\|^{-1}z \notin B_1$ for all $0 \neq z \in Z(\mathfrak{J})$.

We conclude this article with the following example of a unital involutive split quasi Jordan Banach algebra, which is not a JB^*-algebra but its Jordan part is a JB^*-algebra.

Example 5.5 Let J be a JB^*-algebra with norm 1 unit e. Let $\mathfrak{J} = \mathfrak{J} \times \mathfrak{J}$ be the product Banach space with componentwise operations. We define $\triangleleft$ in $\mathfrak{J}$ by $(x, y) \triangleleft (z, w) = (x \circ z, y \circ z)$, where $\circ$ is the Jordan product in J. Then it is easily seen that $(\mathfrak{J}, \triangleleft)$ is a quasi Jordan algebra and $(e, 0)$ is a (right) unit in $\mathfrak{J}$. Moreover, $\mathfrak{J}$ is split with the Jordan part $\{(x, 0) : x \in J\}$ and the zero part $\mathcal{Z}(\mathfrak{J}) = \{(0, y) : y \in J\}$.

Next, we note that the product norm $\|(x, y)\| = \|x\| + \|y\|$ on $\mathfrak{J}$ is complete and satisfies

$$\begin{aligned}\|(x,y) \triangleleft (z,w)\| &= \|(x \circ z, y \circ z)\| \\ &= \|x \circ z\| + \|y \circ z\| \leq \|x\|\|z\| + \|y\|\|z\| \\ &= (\|x\| + \|y\|)\|z\| \leq (\|x\| + \|y\|)(\|z\| + \|w\|) \\ &= \|(x,y)\|\|(z,w)\|,\end{aligned}$$

for all $x, y, z, w \in J$. Hence, $\mathfrak{J}$ is a quasi Jordan Banach algebra, where $(e, 0)$ is a unit with $\|(e, 0)\| = \|e\| + \|0\| = 1$.

Furthermore, there is defined an involution "$*$" on $\mathfrak{J}$ given by $(x, y)^* = (x^*, y^*)$, which is multiplicative (of course, not reverse multiplicative):

$$\begin{aligned}((x,y) \triangleleft (z,w))^* &= (x \circ z, y \circ z)^* \\ &= ((x \circ z)^*, (y \circ z)^*) = (x^* \circ z^*, y^* \circ z^*) \\ &= (x^* \circ z^*, y^* \circ z^*) = (x,y)^* \triangleleft (z,w)^*,\end{aligned}$$

for all $x, y, z, w \in J$. Also, note that

$$\begin{aligned}\|(x,y)^*\| &= \|(x^*, y^*)\| \\ &= \|x^*\| + \|y^*\| = \|x\|\|y\| = \|(x,y)\|,\end{aligned}$$

for all $x, y \in J$. Thus, $\mathfrak{J}$ is a unital involutive split quasi Jordan Banach algebra, which is not a JB^*-algebra; its Jordan part of $\mathfrak{J}$ is isometrically $*$-isomorphic to JB^*-algebra J, and so it satisfies the hypothesis of the above Theorem 5.3.

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