Suppose that $f'''(x)$ exists, with $f(\alpha) = 0$, $f'(\alpha) \neq 0$, and $f''(\alpha) \neq 0$. Show that the rate of convergence of the Newton's method is only quadratic.

For Newton method : $g(x) = x - \dfrac{f(x)}{f'(x)}$

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x) f''(x)}{[f'(x)]^2} \Rightarrow g'(\alpha) = 1 - \frac{[f'(\alpha)]^2 - f(\alpha) f''(\alpha)}{[f'(\alpha)]^2}$$

$$= 1 - \frac{[f'(\alpha)]^2}{[f'(\alpha)]^2} = 1 - 1 = 0$$

$$g''(x) = 0 - \frac{[f'(x)]^2 \{2 f'(x) f''(x) - f'(x) f''(x) - f(x) f'''(x)\} - \{[f'(x)]^2 - f(x) f''(x)\} \cdot 2 f'(x) f''(x)}{[f'(x)]^4}$$

$$g''(\alpha) = - \frac{[f'(\alpha)]^2 \{ f'(\alpha) f''(\alpha) - 0\} - \{[f'(\alpha)]^2 - 0\} \cdot 2 f'(\alpha) f''(\alpha)}{[f'(\alpha)]^4}$$

$$= - \frac{[f'(\alpha)]^3 f''(\alpha) - 2 [f'(\alpha)]^3 f''(\alpha)}{[f'(\alpha)]^4}$$

$$= \frac{[f'(\alpha)]^3 f''(\alpha)}{[f'(\alpha)]^4} = \frac{f''(\alpha)}{f'(\alpha)} \neq 0$$

$\Rightarrow$ Rate of convergence of Newton method is 2 (i.e quadratic) only.