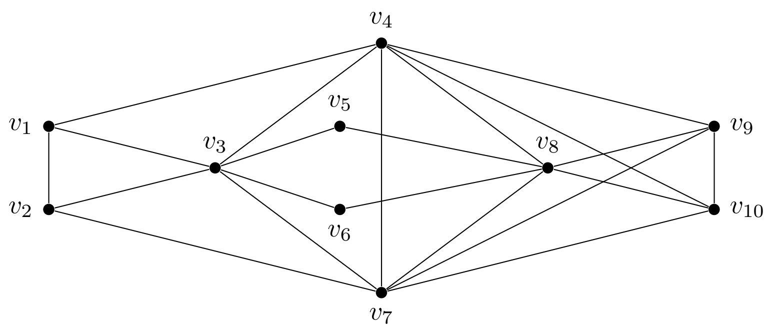
King Saud University, College of Science
Department of Mathematics
Second Semester 1447 H - (2025 – 2026 G)

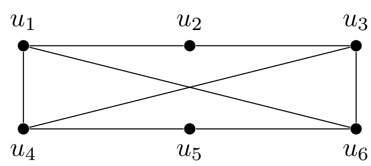
Third Quiz of Math 431.
Allotted time: Half hour

Exercise (5 points)

1. Answer *True* or *False*, justifying each answer.
   (a) The complete bipartite graph $K_{3,3}$ is planar.
   (b) The graph $K_{3,3} - e$ is planar, where e is an edge of $K_{3,3}$, and the graph $K_{3,3} - e$ satisfies Euler's formula.
2. Consider the graphs G and H.



G



H

   (a) Is the graph G planar?
   (b) Show that the chromatic number $\chi(G) \geq 5$.
   (c) Determine the chromatic numbers $\chi(G)$ and $\chi(H)$.

Solution

1. **True/False Justifications:**

   (a) **False.**

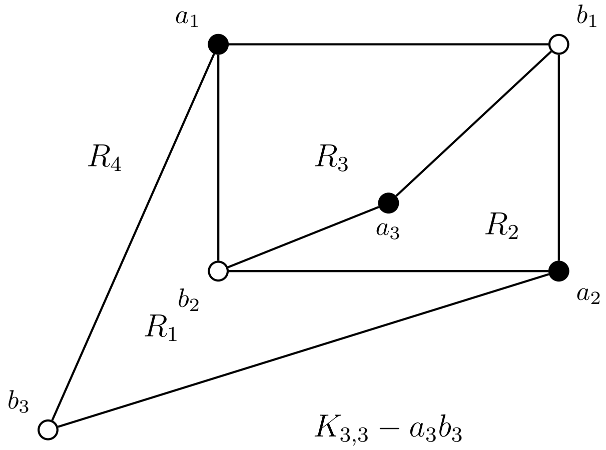
   First justification: According to *Kuratowski's Theorem*, a graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.
   $K_{3,3}$ is one of the two minimal non-planar utility graphs.

   Second justification: Since $K_{3,3}$ is bipartite, it contains no cycles of length 3 (triangles). For any simple planar graph with $V \geq 3$ and without triangle (i.e without C_3), the edge bound is $E \leq 2V - 4$.
   For $K_{3,3}$, we have $V = 6$ and $E = 9$. However, $2(6) - 4 = 8$. Since $9 > 8$, $K_{3,3}$ cannot be planar.

   (b) **True.** Removing any single edge e from $K_{3,3}$ yields a graph with $V = 6$ and $E = 8$. This configuration satisfies the structural planarity bound for triangle-free graphs ($8 \leq 8$). It can be successfully embedded in the plane without intersecting edge paths.

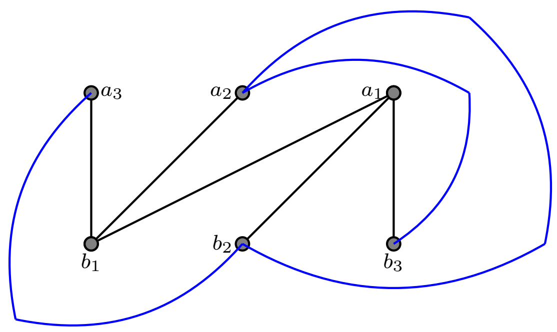
   The complete bipartite graph $K_{3,3}$ is non-planar. However, removing any single edge e makes it planar. Here is a planar drawing where the two sets of vertices are colored differently (black and white) to show the bipartite structure:

   For example: Planar Embedding of $K_{3,3} - a_3b_3$, where $e = a_3b_3$.



   Since $K_{3,3} - e$ is a connected planar graph, it must satisfy *Euler's Formula* $V - E + F = 2$, which gives a face count of exactly $F = 2 - 6 + 8 = 4$ regions (R_1, R_2, R_3 and R_4).

We can represent the bipartite graph $K_{3,3} - a_3b_3$ by another diagram as following:



$K_{3,3} - a_3b_3$

2. **Analysis of Graphs G and H:**

   (a) **Planarity of G: No, G is not planar.** According to *Kuratowski's Theorem*, and since G contains a subgraph $G[\{v_4, v_7, v_8, v_9, v_{10}\}]$ of G that is isomorphic to the complete graph K_5, and K_5 is not a planar graph, then the graph G is not planar.

   (b) **Showing $\chi(G) \geq 5$:** Since G contains a subgraph $G[\{v_4, v_7, v_8, v_9, v_{10}\}]$ of G that is isomorphic to the complete graph K_5, we have $\chi(G) \geq \chi(K_5)$, and $\chi(K_5) = 5$. Therefore, $\chi(G) \geq 5$.

   (c) **Exact Chromatic Numbers:**

   - **For G:** We already established that $\chi(G) \geq 5$. Consider the five colors c_1, c_2, c_3, c_4 and c_5, we can construct a valid 5-coloring for the entire graph as follows:

$$
\begin{array}{lllll}
c(v_4) = c_1, & c(v_7) = c_2, & c(v_8) = c_3, & c(v_9) = c_4 & c(v_{10}) = c_5 \\
c(v_3) = c_4, & c(v_2) = c_1, & c(v_1) = c_2 & & \\
c(v_5) = c_5, & c(v_6) = c_5 & & &
\end{array}
$$

     Since no two adjacent vertices share the same color, this configuration is valid. Therefore, $\chi(G) = 5$.

   - **For H:** It is clear that the graph H is a bipartite. Therefore, $\chi(H) = 2$.