King Saud University, College of Science
Department of Mathematics
QUIZ 1448 H - (2026 – 2027 G)

Second Quiz of Math 431**.**
Allotted time: 30 minutes

Answer just one of the two exercises given.

Question 1

A planar construction and face counting

Let G have vertex set $\{a, b, c, d, x, y\}$. Its edges are the four cycle edges ab, bc, cd, da, together with all eight edges joining a vertex in $\{x, y\}$ to a vertex in $\{a, b, c, d\}$. There are no other edges.

1. Determine the order, size and degree sequence of G. **[2 points]**
2. Construct a plane drawing of G. Your drawing must explain how all edges incident with y can be drawn without crossings. **[3 points]**
3. Determine the number of faces in your drawing, and prove that every face is triangular. **[2 points]**
4. Let G^+ be obtained by adding the edge xy to G. Prove that G^+ is nonplanar. **[2 points]**
5. Is G bipartite? Justify your answer. **[1 points]**

Question 2

Colouring a cycle with a universal vertex

For an integer $r \geq 3$, let $W(r)$ be obtained from the cycle $v_1v_2 \cdots v_rv_1$ by adding one vertex h adjacent to every cycle vertex. Thus $W(r)$ has $r + 1$ vertices.

1. Determine $\chi(W(6))$. Give an explicit optimal colouring and prove that fewer colours cannot suffice. **[3 points]**
2. Determine $\chi(W(5))$. Give an explicit optimal colouring and prove its optimality. **[3 points]**
3. Determine $\chi(W(r))$ for every $r \geq 3$, distinguishing even and odd values of r. Prove your formula. **[3 points]**
4. Describe a plane drawing of $W(r)$. Does your answer to part 2. show that every planar graph is 3-colourable? Explain. **[1 points]**

Solutions

Question 1

1. There are 6 vertices and $4 + 8 = 12$ edges. Each cycle vertex has two cycle neighbours and the neighbours x, y. Both x and y have four neighbours. Thus the degree sequence is

$$\boxed{(4, 4, 4, 4, 4, 4).}$$

2. Start with the induced subgraph

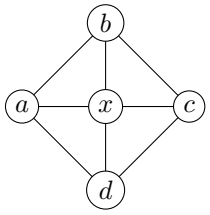
$$G[\{a, b, c, d, x\}] = W(4),$$

where $W(4)$ denotes the wheel obtained from a cycle on four vertices by adding a vertex adjacent to every cycle vertex: the cycle $abcda$ together with the vertex x adjacent to a, b, c, d. Draw the cycle as a diamond and put x inside it, as in Step 1 below.

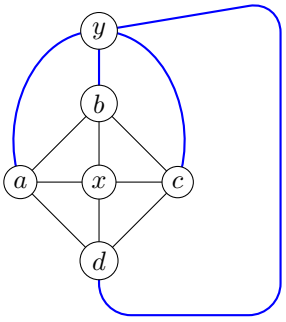
Next, place y outside the cycle and add precisely the four edges

$$ya, \quad yb, \quad yc, \quad yd.$$

Draw all four edges in the exterior of the cycle, as in Step 2. The edge yd follows the outer route around the right-hand side and below the cycle. No edges intersect except at their common endpoints. The resulting drawing is a plane drawing of G.



Step 1. Draw $W(4)$.



Step 2. Add y and its four edges (blue).

3. Euler's formula gives $f = 2 - 6 + 12 = 8$. More explicitly, the faces are

$$xab, \ xbc, \ xcd, \ xda, \ yab, \ ybc, \ ycd, \ yda,$$

where a triple denotes the boundary triangle, not an extra edge or vertex. All eight faces are triangular; in the displayed drawing, yad is the unbounded face.

4. The graph G^+ is simple and has 6 vertices and 13 edges. If it were planar, the planar edge bound would imply $13 \leq 3 \cdot 6 - 6 = 12$, a contradiction. Thus $\boxed{G^+ \text{ is nonplanar.}}$

5. The vertices x, a, b form a triangle. A bipartite graph contains no odd cycle, so G is not bipartite.

Question 2

1. Colour v_1, v_3, v_5 with colour 1, and v_2, v_4, v_6 with colour 2. Give h colour 3. This is a proper colouring, so $\chi(W(6)) \leq 3$. The vertices h, v_1, v_2 induce a triangle, requiring three distinct colours. Hence

$$\boxed{\chi(W(6)) = 3.}$$

2. The cycle induced by $v_1, \ldots, v_5$ is the odd cycle C_5, which cannot be coloured with two colours: alternating colours along a cycle closes properly only when its length is even. Thus the cycle needs at least three colours. Since h is adjacent to every vertex of the cycle, its colour cannot appear on the cycle, so at least four colours are needed.

   An optimal colouring of $(v_1, v_2, v_3, v_4, v_5, h)$ is $(1, 2, 1, 2, 3, 4)$. Therefore

$$\boxed{\chi(W(5)) = 4.}$$

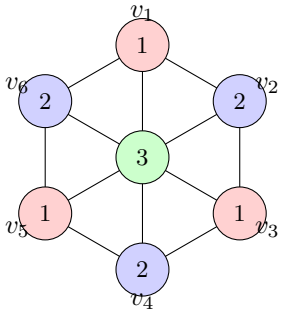
3. If r is even, alternate colours $1, 2$ around the cycle and give h colour 3. A triangle again supplies the matching lower bound.

   If r is odd, the cycle needs at least three colours, and h needs an additional colour. For an explicit four-colouring, alternate $1, 2$ on $v_1, \ldots, v_{r-1}$, give v_r colour 3, and give h colour 4. Consequently

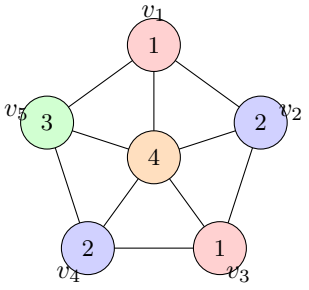
$$\boxed{\chi(W(r)) = \begin{cases} 3, & r \text{ even}, \\ 4, & r \text{ odd}. \end{cases}}$$

   This includes $r = 3$, for which $W(3) = K_4$.

4. Draw the cycle as a regular polygon, put h at its centre, and join h to its vertices by straight segments. No edges intersect except at their common endpoints. Thus every $W(r)$ is planar. Part 2. *disproves* the statement that every planar graph is 3-colourable: $W(5)$ is a planar graph with chromatic number 4.



$W(6)$: $\chi = 3$



$W(5)$: $\chi = 4$

Numbers inside vertices denote colours; the central vertex is h.