King Saud University, College of Science
Department of Mathematics
Second Semester 1447 H - (2025 – 2026 G)

Second Quiz of Math 431.
Allotted time: Half hour

Exercise (5 points)
Consider the following two sequences of integers:

$$D_1 = (1, 1, 2, 2, 3, 5, 6, 6), \qquad D_2 = (1, 1, 3, 3, 4, 4, 6, 6).$$

1. (a) Determine whether there exists a *graph* G_1 whose degree sequence is D_1.
   (b) Prove that D_1 is not graphical; that is, show that there is no simple graph whose degree sequence is D_1.
2. (a) Determine whether there exists a *simple bipartite graph* G_2 whose degree sequence is D_2.
   (b) Prove that D_2 is graphical and construct explicitly a simple graph G whose degree sequence is D_2.

Solution.

1. The sequence $D_1 = (1, 1, 2, 2, 3, 5, 6, 6)$.

(a) Determine whether there exists a graph G_1 whose degree sequence is D_1.

The sum of the degrees is

$$1 + 1 + 2 + 2 + 3 + 5 + 6 + 6 = 26.$$

By the Handshaking Lemma, the sum of the degrees of a graph equals $2|E|$, hence must be even. Since 26 is even, this necessary condition is satisfied.

If multiple edges and loops are allowed, this condition is also sufficient: every finite sequence of non-negative integers with even sum is the degree sequence of some graph (possibly with loops and multiple edges).

Therefore, there exists a graph G_1 whose degree sequence is D_1.

Construction of a non-simple graph with degree sequence $D_1 = (1, 1, 2, 2, 3, 5, 6, 6)$.

Arrange the sequence in non-increasing order:

$$(6, 6, 5, 3, 2, 2, 1, 1).$$

Let

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$$

with prescribed degrees

$$\deg(v_1) = 6,\ \deg(v_2) = 6,\ \deg(v_3) = 5,\ \deg(v_4) = 3,\ \deg(v_5) = 2,\ \deg(v_6) = 2,\ \deg(v_7) = 1,\ \deg(v_8) = 1.$$

We construct an example of a graph allowing multiple edges as follows.

- Add three parallel edges between v_1 and v_2.
- Add the edges
$$v_1v_3,\ v_1v_4,\ v_1v_5.$$
- Add the edges
$$v_2v_3,\ v_2v_4,\ v_2v_6.$$
- Add the edges
$$v_3v_4,\ v_3v_7,\ v_3v_8.$$
- Add the edge
$$v_5v_6.$$

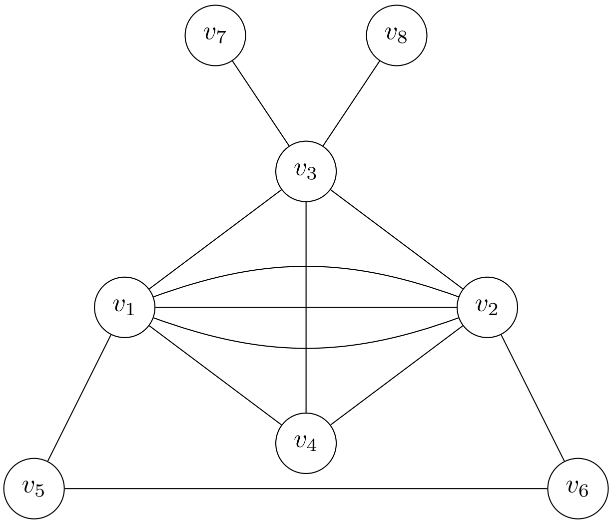
Verification of degrees:

$$\begin{aligned}
\deg(v_1) &= 3 + 1 + 1 + 1 = 6,\\
\deg(v_2) &= 3 + 1 + 1 + 1 = 6,\\
\deg(v_3) &= 1 + 1 + 1 + 1 + 1 = 5,\\
\deg(v_4) &= 1 + 1 + 1 = 3,\\
\deg(v_5) &= 1 + 1 = 2,\\
\deg(v_6) &= 1 + 1 = 2,\\
\deg(v_7) &= 1,\\
\deg(v_8) &= 1.
\end{aligned}$$

Thus the resulting graph has degree sequence

$$(6, 6, 5, 3, 2, 2, 1, 1),$$

which is exactly D_1.



(b) We now apply the Havel–Hakimi algorithm.

First arrange the sequence in non-increasing order:

$$(6, 6, 5, 3, 2, 2, 1, 1).$$

Remove the first 6 and subtract 1 from the next six terms:

$$(5, 4, 2, 1, 1, 0, 1).$$

Reorder:

$$(5, 4, 2, 1, 1, 1, 0).$$

Remove 5 and subtract 1 from the next five terms:

$$(3, 1, 0, 0, 0, 0).$$

Reorder:

$$(3, 1, 0, 0, 0, 0).$$

Remove 3 and subtract 1 from the next three terms:

$$(0, -1, -1, 0, 0).$$

Since negative entries appear, the sequence is not graphical.

Therefore, D_1 is not the degree sequence of any simple graph.

2. The sequence $D_2 = (1, 1, 3, 3, 4, 4, 6, 6)$.

(a) In a simple bipartite graph on 8 vertices, the maximum possible degree of a vertex is at most the size of the opposite part.

Since D_2 contains vertices of degree 6, each such vertex must be adjacent to 6 distinct vertices. In a bipartite graph, this forces the opposite part to contain at least 6 vertices. But then the remaining part would contain at most 2 vertices, making it impossible for two vertices to both have degree 6.

Hence, no simple bipartite graph has degree sequence D_2.

(b) We now show that D_2 is graphical using the Havel–Hakimi algorithm.

Arrange in non-increasing order:

$$(6, 6, 4, 4, 3, 3, 1, 1).$$

Remove 6 and subtract 1 from the next six terms:

$$(5, 3, 3, 2, 2, 0, 1).$$

Reorder:

$$(5, 3, 3, 2, 2, 1, 0).$$

Remove 5 and subtract 1 from the next five terms:

$$(2, 2, 1, 1, 0, 0).$$

Reorder:

$$(2, 2, 1, 1, 0, 0).$$

Remove 2 and subtract 1 from the next two terms:

$$(1, 0, 1, 0, 0).$$

Reorder:

$$(1, 1, 0, 0, 0).$$

Remove 1 and subtract 1 from the next term:

$$(0, 0, 0, 0).$$

Since we arrive at the zero sequence, D_2 is graphical.

Construction of a graph.

Label the vertices $v_1, \ldots, v_8$ corresponding initially to degrees

$$(6, 6, 4, 4, 3, 3, 1, 1).$$

Following the reductions above, one possible realization is obtained by joining:

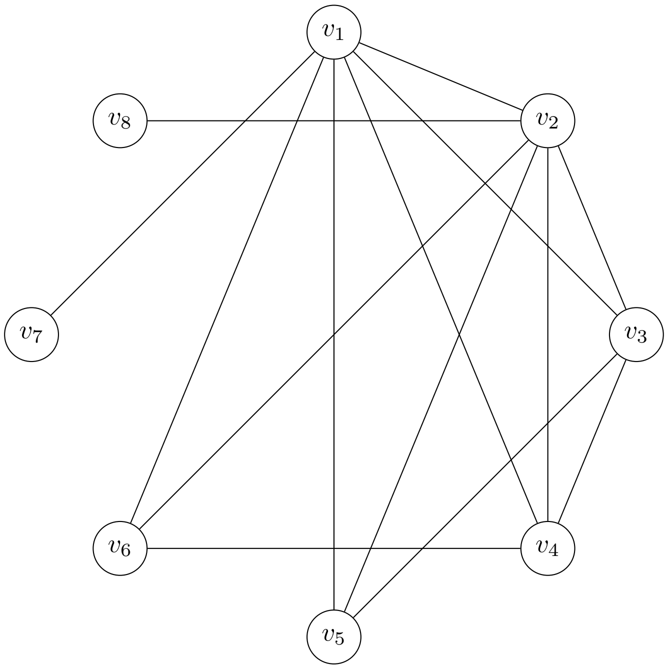
$$\begin{aligned}
&v_1 \text{ to } v_2, v_3, v_4, v_5, v_6, v_7, \\
&v_2 \text{ to } v_3, v_4, v_5, v_6, v_8, \\
&v_3 \text{ to } v_4, v_5, \\
&v_4 \text{ to } v_6.
\end{aligned}$$

It is straightforward to verify that the degrees are

$$(6, 6, 4, 4, 3, 3, 1, 1).$$

Thus D_2 is graphical.

Here is an example of a simple graph G whose degree sequence is D_2:



v_1
v_8
v_2
v_7
v_3
v_6
v_4
v_5