STAT 216
Maram Alomari

Practice Problems

Discrete Random Variable

- Binomial Distribution
- Poisson Distribution
- Other Discrete Distribution

Problem 18.17

A package of 6 fuses are tested where the probability an individual fuse is defective is 0.05. (That is, 5% of all fuses manufactured are defective).
(a) What is the probability one fuse will be defective?
(b) What is the probability at least one fuse will be defective?
(c) What is the probability that more than one fuse will be defective, given at least one is defective?

Binomial distribution is used to model how many times an event happens in a series of n trials, each trial is independent and has only two outcomes $\begin{cases} p \\ 1-p \end{cases}$

a) $P(X=1) = \binom{6}{1}(0.05)(0.95)^5 \Rightarrow$

b) $P(X \geq 1) = 1 - P(X<1) \Rightarrow 1 - P(0) \Rightarrow 1 - (0.95)^6 =$

c) $P(X>1 \mid X \geq 1) = \dfrac{P(X>1)}{P(X\geq 1)} = \dfrac{1-P(X\leq 1)}{1-P(X<1)} = \dfrac{1-P(0)-P(1)}{1-P(0)} =$

$$p(r) = C(n,r)p^r q^{n-r} \qquad E(X) = np \qquad Var(X) = np(1-P)$$

Problem 18.14 $p = 1/6$ n

If X is the number of "6"'s that turn up when 72 ordinary dice are independently thrown, find the expected value of X^2.

$E(x^2)$

$var(x) = E(x^2) - E(x)^2 \Rightarrow E(x^2) = Var(x) + E(x)^2$

$= 10 + 144$

$= 154$

$var(X) = npq$

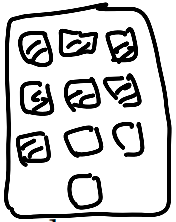
$= 72(1/6)(5/6)$

$= 10$

$E(x)^2 = (np)^2 \Rightarrow (72(1/6))^2 = 144$

$$p(r) = C(n,r)p^r q^{n-r} \qquad E(X) = np \qquad Var(X) = np(1-P)$$

Binomial Distribution



There are 3 seats left and 5 possible passengers

X : number of passengers with 50% chance of showing up

Problem 18.9

Suppose an airline accepted 12 reservations for a commuter plane with 10 seats. They know that 7 reservations went to regular commuters who will show up for sure. The other 5 passengers (n) will show up with a 50% (p) chance, independently of each other.

(a) Find the probability that the flight will be overbooked. ⇒ meaning more than 3 will show up

(b) Find the probability that there will be empty seats. ⇒ less than 3 will show up

$$a)\ P(X>3) = P(X=4) + P(X=5) \Rightarrow \binom{5}{4}(0.5)^4(0.5) + \binom{5}{5}(0.5)^5 = 0.187$$

$$b)\ P(X<3) = 1 - P(X\geq 3) \Rightarrow 1 - P(X=3) - P(X>3)$$

$$= 1 - 0.3125 - 0.187$$

$$= 0.5$$

$$p(r) = C(n,r)p^r q^{n-r} \qquad E(X) = np \qquad Var(X) = np(1-P)$$

Problem 18.7 ‡

A company prices its hurricane insurance using the following assumptions:

each year has two outcomes → No hurricane / → one hurricane

(i) In any calendar year, there can be at most one hurricane.

(ii) In any calendar year, the probability of a hurricane is 0.05 . p

(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.

Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period.

n

$$P(X<3) = P(X=0) + P(X=1) + P(X=2)$$

$$= \binom{20}{0}(0.95)^{20} + \binom{20}{1}(0.05)(0.95)^{19} + \binom{20}{2}(0.05)^{2}(0.95)^{18}$$

$$= 0.9245$$

$$p(r) = C(n,r)p^r q^{n-r} \qquad E(X) = np \qquad Var(X) = np(1-P)$$

C : the event of receiving the shipment from Company X

I : the event of ineffective vials

Problem 18.4 ‡

A hospital receives 1/5 of its flu vaccine shipments from Company X and the remainder of its shipments from other companies. Each shipment contains a very large number of vaccine vials.

For Company Xs shipments, 10% of the vials are ineffective. For every other company, 2% of the vials are ineffective. The hospital tests 30 randomly selected vials from a shipment and finds that one vial is ineffective. What is the probability that this shipment came from Company X?

$$\boxed{\text{The event } I \text{ depends on the event } C}$$

we have :

$$p(I|C) = \binom{30}{1}(0.1)(0.9)^{29}$$

$$p(I|C') = \binom{30}{1}(0.02)(0.98)^{29}$$

$$p(C) = 1/5 \qquad p(C') = 4/5$$

we need to calculate :

$$p(C|I) \,?$$

Since we have conditional probabilities and we want to find the reverse we use Bayes' Formula

$$p(C|I) = \frac{p(I|C)\,p(C)}{p(I|C)\,p(C) + p(I|C')\,p(C')}$$

$$= \frac{(0.141)(\frac{1}{5})}{0.141(\frac{1}{5}) + 0.334(\frac{4}{5})} = 0.096$$

$$p(r) = C(n,r)p^r q^{n-r} \qquad E(X) = np \qquad Var(X) = np(1-P)$$

Problem 19.1
Suppose the average number of car accidents on the highway in one day is 4. What is the probability of no car accident in one day? What is the probability of 1 car accident in two days?

poisson distribution is used to model the occurrance of an event in a specefic range.
Ex: number of accidents in a month

$X \sim poisson(4)$ $\quad p(X) = \frac{e^{-4} 4^{x}}{x!}$

① $p(X=0) = e^{-4} = 0.0183$

② $X \sim piosson(8)$

$p(X=1) = e^{-8} \cdot 8 = 0.00268$

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad E(X) = \lambda \qquad Var(X) = \lambda$$

Problem 19.5

At a checkout counter customers arrive at an average of 2 per minute. Find the probability that

(a) at most 4 will arrive at any given minute $\lambda=2$

(b) at least 3 will arrive during an interval of 2 minutes $\lambda=4$

(c) 5 will arrive in an interval of 3 minutes. $\lambda=6$

a) $P(X\leq 4) = P(0)+P(1)+P(2)+P(3)+P(4) = \sum_{x=0}^{4} \frac{e^{-2}\cdot 2^{x}}{x!} = 0.947$

b) $P(X\geq 3) = 1-P(X<3) = 1-P(0)-P(1)-P(2) = 0.762$

c) $P(X=5) = \frac{e^{-6}\cdot 6^{5}}{5!} = 0.161$

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad E(X) = \lambda \qquad Var(X) = \lambda$$

Problem 19.14

Let X be a Poisson random variable with mean λ. If $P(X = 1|X \leq 1) = 0.8$, what is the value of λ?

$$P(X=1 | X \leq 1) = \frac{P(X=1)}{P(X \leq 1)} = \frac{e^{-\lambda}\lambda}{e^{-\lambda} + e^{-\lambda}\lambda}$$

$$\Rightarrow \frac{\cancel{e^{-\lambda}} \cdot \lambda}{\cancel{e^{-\lambda}}(1+\lambda)} = \frac{\lambda}{1+\lambda} = 0.8$$

$$\Rightarrow \lambda = 0.8 + 0.8\lambda \Rightarrow 0.2\lambda = 0.8 \Rightarrow \lambda = 4$$

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad E(X) = \lambda \qquad Var(X) = \lambda$$

Problem 19.6

Suppose the number of hurricanes in a given year can be modeled by a random variable having Poisson distribution with standard deviation $\sigma = 2$. What is the probability that there are at least three hurricanes?

$\lambda = \sigma^2 \Rightarrow \lambda = 4$

$p(X \geq 3) = 1 - p(0) - p(1) - p(2)$

$= 0.7618$

$p(k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad E(X) = \lambda \qquad Var(X) = \lambda$

Problem 20.1

An urn contains 5 white, 4 black, and 1 red marble. Marbles are drawn, with replacement, until a red one is found. If X is the random variable counting the number of trials until a red marble appears, then
(a) What is the probability that the red marble appears on the first trial?
(b) What is the probability that the red marble appears on the second trial?
(c) What is is the probability that the marble appears on the kth trial.

We use Geomatric distribution to model the number of trials needed until the event occures.

Ex. how many times a coin is tossed until we see a head

$P(R) = \frac{1}{10}$ $\quad P(R`) = \frac{9}{10}$

a) $\frac{1}{10} = 0.1$

b) $\left(\frac{9}{10}\right)\left(\frac{1}{10}\right) = 0.09$

c) $\left(\frac{9}{10}\right)^{k-1}\left(\frac{1}{10}\right)$

$$p(n) = pq^{n-1}, n = 1,2,3, \ldots . \qquad E(X) = p^{-1} \qquad Var(X) = \frac{1-p}{p^2}$$

Problem 20.6

Suppose one die is rolled over and over until a Two is rolled. What is the probability that it takes from 3 to 6 rolls?

$\frac{1}{6}$

$$p(3 \le X \le 6) = p(3) + p(4) + p(5) + p(6)$$

$$= \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right)$$

$$= 0.359$$

$$p(n) = pq^{n-1}, n = 1,2,3, \ldots. \qquad E(X) = p^{-1} \qquad Var(X) = \frac{1-p}{p^2}$$

Problem 20.12

An actress has a probability of getting offered a job after a try-out of $p = 0.10$. She plans to keep trying out for new jobs until she gets offered. Assume outcomes of try-outs are independent.

(a) How many try-outs does she expect to have to take?

(b) What is the probability she will need to attend more than 2 try-outs?

a) $E(X) = p^{-1} = 0.10^{-1} = 10$

b) $p(X > 2) = 1 - p(X \leq 2)$

$= 1 - p(1) - p(2)$

$= 1 - (0.1) - (0.9)(0.1) = 0.81$

$p(n) = pq^{n-1}, n = 1,2,3, \dots.$ $\quad E(X) = p^{-1}$ $\quad Var(X) = \dfrac{1-p}{p^2}$

Problem 20.11 ‡

As part of the underwriting process for insurance, each prospective policyholder is tested for high blood pressure. Let X represent the number of tests completed when the first person with high blood pressure is found. The expected value of X is 12.5.

Calculate the probability that the sixth person tested is the first one with high blood pressure.

$E(X) = 12.5 \Rightarrow p = 1/12.5 = 0.08$

$p(X=6) = (0.92)^5(0.08)$

$= 0.053$

$p(n) = pq^{n-1}, n = 1,2,3,\ldots.$ $\quad E(X) = p^{-1}$ $\quad Var(X) = \dfrac{1-p}{p^2}$

Problem 20.15

A geological study indicates that an exploratory oil well drilled in a particular region should strike oil with $p = 0.2$. Find P(the 3rd oil strike comes on the 5th well drilled).

$$P(X=3) = \binom{4}{2}(0.2)^{3}(0.8)^{5-3}$$

$$= 6(0.2)^{3}(0.8)^{2}$$

$$= 0.03$$

Negative Binomial is used to model how many times an event occurs in a series of trials

EX: getting 3 heads when we toss the coin 7 times

$$p(r) = C(n-1, r-1)p^{r}q^{n-r} \qquad E(X) = rp^{-1} \qquad Var(X) = \frac{r(1-p)}{p^2}$$

Problem 20.17

Find the expected value and the variance of the number of times one must throw a die until the outcome 1 has occurred 4 times.

$p = 1/6$ $\quad$ r

$$E(X) = rp^{-1} = 4\left(\frac{1}{6}\right)^{-1} = 24$$

$$Var(X) = \frac{r(1-p)}{p^2} = \frac{4\left(\frac{5}{6}\right)}{\left(\frac{1}{6}\right)^2} = 120$$

$$p(r) = C(n-1, r-1)p^r q^{n-r} \qquad E(X) = rp^{-1} \qquad Var(X) = \frac{r(1-p)}{p^2}$$

Problem 20.18

$p=0.5$ $r=4$

Find the probability that a man flipping a coin gets the fourth head on the ninth flip.

$n=9$

$$p(X=4) = \binom{8}{3}(0.5)^9$$

$$= 0.109$$

$$p(r) = C(n-1, r-1)p^r q^{n-r} \qquad E(X) = rp^{-1} \qquad Var(X) = \frac{r(1-p)}{p^2}$$

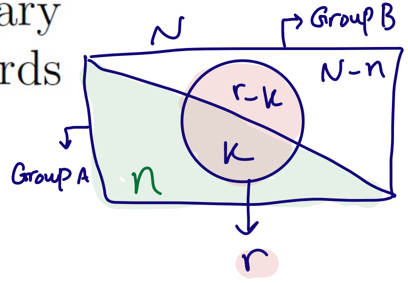
Problem 20.27

Suppose we randomly select 5 cards without replacement from an ordinary deck of playing cards. What is the probability of getting exactly 2 red cards (i.e., hearts or diamonds)?

r

K

52 card



selection from red cards

selection from black cards

$$p(X=2) = \frac{\binom{26}{2}\binom{26}{3}}{\binom{52}{5}} = 0.3251$$

the whole selection

When we have a population divided into two groups (A, B) and we select a sample without replacement we use hypergeomatric to model how many we select from group A

$$p(k) = \frac{C(n,k)C(N-n,r-k)}{C(N,r)} \qquad E(X) = \frac{nr}{N} \qquad Var(X) = \frac{nr}{N}\left[\frac{(n-1)(r-1)}{N-1} + 1 - \frac{nr}{N}\right]$$

Problem 20.32

A package of 8 AA batteries contains 2 batteries that are defective. A student randomly selects four batteries and replaces the batteries in his calculator.
(a) What is the probability that all four batteries work?
(b) What are the mean and variance for the number of batteries that work?

N-n

6

a) $p(X=4) = \dfrac{\binom{6}{4}\binom{2}{0}}{\binom{8}{4}} = 0.214$

b) $E(X) = \dfrac{6 \times 4}{8} = 3$

Group A

sample

the population

$Var(X) = 3\left[\dfrac{5 \times 3}{7} + 1 - 3\right]$

$= 0.42$

$$p(k) = \frac{C(n,k)C(N-n,r-k)}{C(N,r)} \qquad E(X) = \frac{nr}{N} \qquad Var(X) = \frac{nr}{N}\left[\frac{(n-1)(r-1)}{N-1} + 1 - \frac{nr}{N}\right]$$

Problem 20.35

Consider an urn with 7 red balls and 3 blue balls. Suppose we draw 4 balls without replacement and let X be the total number of red balls we get. Compute $P(X \leq 1)$.

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$\frac{\binom{7}{0}\binom{3}{4}}{\binom{10}{4}} \Rightarrow P(X=1) = \frac{\binom{7}{1}\binom{3}{3}}{\binom{10}{4}} = 0.033$$

we can't choose 4 from 3

$\therefore p(0) = 0$

$$p(k) = \frac{C(n,k)C(N-n, r-k)}{C(N,r)} \qquad E(X) = \frac{nr}{N} \qquad Var(X) = \frac{nr}{N}\left[\frac{(n-1)(r-1)}{N-1} + 1 - \frac{nr}{N}\right]$$

Problem 20.38 n

A box contains 10 white and 15 black marbles. Let X denote the number of white marbles in a selection of 10 marbles selected at random and without replacement. Find $\frac{\text{Var}(X)}{E(X)}$. r

$$\frac{Var(X)}{E(X)} = \frac{\cancel{\frac{nr}{N}}\left[\frac{n-1(r-1)}{N-1} + 1 - \frac{nr}{N}\right]}{\cancel{\frac{nr}{N}}}$$

$$= \frac{9\times 9}{24} + 1 - 4 \quad = 0.375$$

$$p(k) = \frac{C(n,k)C(N-n,r-k)}{C(N,r)} \qquad E(X) = \frac{nr}{N} \qquad Var(X) = \frac{nr}{N}\left[\frac{(n-1)(r-1)}{N-1} + 1 - \frac{nr}{N}\right]$$