STAT 216
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Practice Problems

Continuous Random Variables

- > Distribution Functions
- > Expectation, Variance and Standard Deviation
- > The Uniform Distribution Function
- > Normal Random Variables
- > Exponential and Gamma Distributions

Problem 22.1

Determine the value of c so that the following function is a pdf.

$$f(x) = \begin{cases} \frac{c}{(x+1)^3} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$c \int_0^\infty (x+1)^{-3}\, dx = 1$$

$$c \left. \frac{(x+1)^{-2}}{-2} \right|_0^\infty = 1$$

$$c\left(\frac{1}{2}\right) = 1 \quad \Rightarrow \quad c = 2$$

- $f(x)$ must be nonnegative for each value of the random variable.
- The integral over all values of the random variable must equal one.

Problem 22.3

A college professor never finishes his lecture before the bell rings to end the period and always finishes his lecture within ten minutes after the bell rings. Suppose that the time X that elapses between the bell and the end of the lecture has a probability density function

$$f(x) = \begin{cases} kx^2 & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

where $k > 0$ is a constant.
Determine the value of k and find the probability the lecture ends less than 3 minutes after the bell.

$$k \int_0^{10} x^2 \, dx = 1 \quad \Rightarrow \quad k \frac{x^3}{3} \Big|_0^{10} = 1 \quad \Rightarrow \quad k = \frac{3}{1000} = 0.003$$

$$P(X < 3) = \int_0^3 0.003 x^2 \, dx = 0.003 \left(\frac{3^3}{3} \right) = 0.027$$

- $f(x)$ must be nonnegative for each value of the random variable.
- The integral over all values of the random variable must equal one.

Problem 22.5

Define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 1 \\ (x+2)/6 & 1 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

we need to make sure that there are no gaps between the functions over the interval then we can say that F(x) is cont.

(a) Is F the cdf of a continuous random variable? Explain your answer.
(b) If your answer to part (a) is "yes", determine the corresponding pdf; if your answer was "no", then make a modification to F so that it is a cdf, and then compute the corresponding pdf.

a) we chick continuity on the points where the function changed

$0: \quad 0 = \frac{0}{2}$ ✓

$1: \quad \frac{1}{2} = \frac{3}{6}$ ✓

$4: \quad \frac{6}{6} = 1$ ✓

∴ Yes, the cdf is continuous

b) $f(x) = F'(x)$

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{1}{6} & 1 \le x < 4 \\ 0 & x \geq 4 \end{cases}$$

- $f(x)$ must be nonnegative for each value of the random variable.
- The integral over all values of the random variable must equal one.

Problem 22.6

The amount of time it takes a driver to react to a changing stoplight varies from driver to driver, and is (roughly) described by the continuous probability function of the form:

$$f(x) = \begin{cases} kxe^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant and x is measured in seconds.
(a) Find the constant k.
(b) Find the probability that a driver takes more than 1 second to react.

- $f(x)$ must be nonnegative for each value of the random variable.
- The integral over all values of the random variable must equal one.

Problem 23.1

Let X have the density function given by

$$f(x) = \begin{cases} 0.2 & -1 < x \leq 0 \\ 0.2 + cx & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c.
(b) Find $F(x)$.
(c) Find $P(0 \leq x \leq 0.5)$.
(d) Find $E(X)$.

$$E\big(g(x)\big) = \int_{-\infty}^{\infty} g(x) f(x) dx \qquad Var(X) = E\left(X^2\right) - E(X)^2$$

Problem 23.2

The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose also that you are told that $E(X) = \frac{3}{5}$.

(a) Find a and b.

(b) Determine the cdf, $F(x)$, explicitly. $= \int_0^x f(t)\,dt$

a) $\int_0^1 a + bx\,dx = 1 \Rightarrow ax + \frac{bx^3}{3}\Big|_0^1 = 1$

$\Rightarrow a + \frac{b}{3} = 1$ (1)

$\int_0^1 ax + bx^3\,dx = \frac{3}{5} \Rightarrow \frac{ax^2}{2} + \frac{bx^4}{4}\Big|_0^1 = \frac{3}{5}$

$\Rightarrow \left[\frac{a}{2} + \frac{b}{4} = \frac{3}{5}\right] \times 2 \Rightarrow a + \frac{b}{2} = \frac{6}{5}$ (2)

(1) − (2)

$$\boxed{\begin{aligned} &a + \frac{b}{2} = \frac{6}{5} \\ -\;&a + \frac{b}{3} = 1 \\ \hline &0 + \frac{b}{6} = \frac{1}{5} \end{aligned}}$$

$\Rightarrow b = \frac{6}{5}$

plug it in (1)

$a = \frac{3}{5}$

b) $F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x a + bt^2\,dt = at + \frac{bt^3}{3}\Big|_0^x = ax + \frac{bx^3}{3}, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx \qquad Var(X) = E(X^2) - E(X)^2$$

Problem 23.7
Let X have a cdf

$$F(x) = \begin{cases} 1 - \frac{1}{x^6} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find Var(X).

$$f(x) = 6x^{-7}$$

$$Var(X) = \int_1^\infty x^2 \cdot 6x^{-7}\,dx - \left(\int_1^\infty x \cdot 6x^{-7}\,dx\right)^2$$

$$= -\frac{6x^{-4}}{4}\Big|_1^\infty - \left(-6\frac{x^{-5}}{5}\Big|_1^\infty\right)^2$$

$$= \frac{6}{4} - \left(\frac{6}{5}\right)^2 = 0.06$$

$$E(g(x)) = \int_{-\infty}^{\infty} g(x)f(x)dx \qquad Var(X) = E(X^2) - E(X)^2$$

Uniform

Problem 24.1

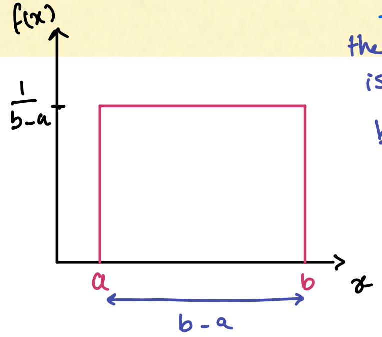
The total time to process a loan application is uniformly distributed between 3 and 7 days.

(a) Let X denote the time to process a loan application. Give the mathematical expression for the probability density function.

(b) What is the probability that a loan application will be processed in fewer than 3 days ?

(c) What is the probability that a loan application will be processed in 5 days or less ?

The uniform distribution has a constant pdf over an interval (a,b).
Since we have a constant function the distribution graph is a rectangular



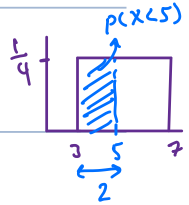
$\int_a^b f(x)\,dx = 1$ the whole area is equal to 1

$\underbrace{b-a}_{\text{width}} \times \underbrace{\frac{1}{b-a}}_{\text{hight}} = 1$

a) $f(X) = \frac{1}{b-a} = \frac{1}{7-3} = \frac{1}{4}$

b) $p(X<3) = F(3) = \frac{3-3}{4} = 0$

c) $p(X<5) = F(5) = \frac{5-3}{4} = \frac{2}{4} = 0.5$



$2 \times \frac{1}{4} = 0.5$

$$f(x) = \frac{1}{b-a} \qquad E(X) = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12} \qquad F(x) = \frac{x-a}{b-a}$$

Problem 24.2

Customers at TAB are charged for the amount of salad they take. Sampling suggests that the amount of salad taken is uniformly distributed between 5 a ounces and b 15 ounces. Let X = salad plate filling weight

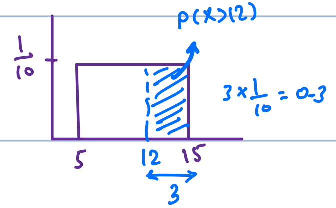
(a) Find the probability density function of X.

(b) What is the probability that a customer will take between 12 and 15 ounces of salad?

(c) Find $E(X)$ and Var(X).

a) $f(x) = \dfrac{1}{b-a} = \dfrac{1}{15-5} = \dfrac{1}{10}$

b) $P(12 < X < 15) = P(X > 12) = \dfrac{15-12}{10} = 0.3$

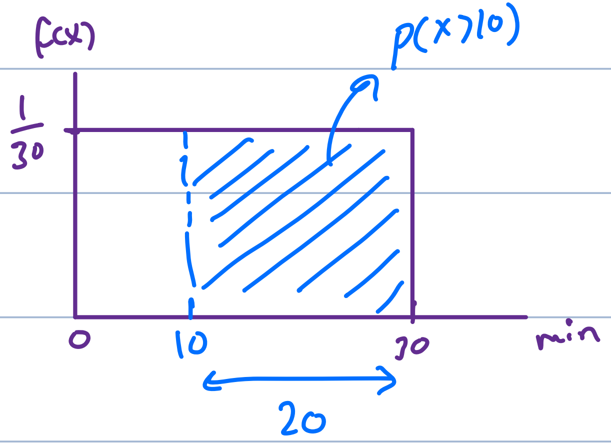


c) $E(X) = \dfrac{20}{2} = 10$, $Var(X) = \dfrac{10^2}{12} =$

$$f(x) = \frac{1}{b-a} \qquad E(X) = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12} \qquad F(x) = \frac{x-a}{b-a}$$

Problem 24.6

You arrive at a bus stop at 10:00 am, knowing that the bus will arrive at some time uniformly distributed between 10:00 and 10:30. What is the probability that you will have to wait longer than 10 minutes?



$20 \times \frac{1}{30} =$

$$P(X > 10) = \frac{30-10}{30}$$

$$f(x) = \frac{1}{b-a} \qquad E(X) = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12} \qquad F(x) = \frac{x-a}{b-a}$$

Problem 24.11

Let X be a random variable with a continuous uniform distribution on the interval $(1, a), a > 1$. If $E(X) = 6\text{Var}(X)$, what is the value of a?

$$E(X) = \frac{1+a}{2} \qquad Var(X) = \frac{(a-1)^2}{12}$$

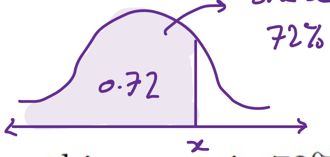
$$\frac{1+a}{\cancel{2}} = \cancel{6}\,\frac{(a-1)^2}{\cancel{2}\,\cancel{12}} \Rightarrow 1+a = (a-1)^2 \Rightarrow \cancel{1+a} = (a-1)\cancel{(a+1)}$$

$$\Rightarrow a - 1 = 1 \Rightarrow a = 2$$

$$f(x) = \frac{1}{b-a} \qquad E(X) = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12} \qquad F(x) = \frac{x-a}{b-a}$$

Problem 25.1
Scores for a particular standardized test are normally distributed with a mean of 80 (μ) and a standard deviation of 14 (σ). Find the probability that a randomly chosen score is
(a) no greater than 70
(b) at least 95
(c) between 70 and 95.
(d) A student was told that her percentile score on this exam is 72%. Approximately what is her raw score?

Normal distribution can be transformed to standard normal z
$Z = \frac{x-\mu}{\sigma}$
Then we use
z table to find the probability



She scored higher than 72% of the class
(72% of the scores are less than hers)

a) $P(X<70) = P\left(Z < \frac{70-80}{14}\right) = \Phi(-0.714) = 1-\Phi(0.714) = 0.2389$

b) $P(X>95) = P\left(Z > \frac{95-80}{14}\right) = 1-\Phi(1.07) = 0.1423$

c) $P(70<X<95) = P(-0.71 < Z < 1.07) = \Phi(1.07) - \Phi(-0.71) = 0.6188$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

$$P(X \le a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right).$$

d) $p(X < x) = 0.72$

Area under the Standard Normal Curve from $-\infty$ to x

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Since we don't have an exact value, we choose the closest one.

0.719 is less by 0.001

0.7224 is greater by 0.0024

$\Rightarrow$ 0.719 is closer

$$p\left(Z < \frac{x-80}{14}\right) = 0.72$$

$$\frac{x-80}{14} = 0.58 \Rightarrow x = 0.58 \times 14 + 80$$

$$x = 88.12 \Rightarrow \text{approximately } 88$$

Normal

Problem 25.2

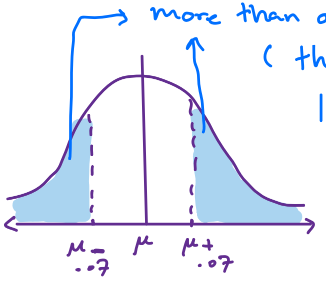
Suppose that egg shell thickness is normally distributed with a mean of 0.381 mm and a standard deviation of 0.031 mm.

(a) Find the proportion of eggs with shell thickness more than 0.36 mm.

(b) Find the proportion of eggs with shell thickness within 0.05 mm of the mean.

(c) Find the proportion of eggs with shell thickness more than 0.07 mm from the mean.

$\mu \pm 0.07$



more than 0.07 from the mean

(the area where the |difference| between X and μ more than 0.07)

a) $p(X > 0.36) = p\left(Z > \dfrac{0.36 - 0.381}{0.031}\right)$

b) $p\left(\dfrac{(\cancel{\mu} - 0.05) - \cancel{\mu}}{\sigma} < Z < \dfrac{(\cancel{\mu} + 0.05) - \cancel{\mu}}{\sigma}\right) = p\left(\dfrac{-.05}{.031} < Z < \dfrac{.05}{.031}\right)$

c) $p(X < \mu - 0.07) + p(X > \mu + 0.07) = p\left(Z < \dfrac{-0.07}{0.031}\right) + p\left(Z > \dfrac{0.07}{0.031}\right)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

$$P(X \le a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Problem 25.3

Assume the time required for a certain distance runner to run a mile follows a normal distribution with mean 4 minutes and variance 4 seconds.

(a) What is the probability that this athlete will run the mile in less than 4 minutes?

(b) What is the probability that this athlete will run the mile in between 3 min55sec and 4min5sec?

the units of μ and σ^2 are different

either $\mu = 4 \times 60$ sec

or $\sigma^2 = 4/60$ min

a) $p(X < 4) = p\left(Z < \frac{4-4}{\sqrt{4/60}}\right) = p(Z < 0) = 0.5$

b) $X = 3\text{ min } 55\text{ Sec} \Rightarrow 3 + \frac{55}{60} = 3.92$

$X = 4\text{ min } 5\text{ sec} \Rightarrow 4 + \frac{5}{60} = 4.08$

$p\left(\frac{3.92-4}{\sqrt{4/60}} < Z < \frac{4.08-4}{\sqrt{4/60}}\right)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

$$P(X \le a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Problem 25.4

You work in Quality Control for GE. Light bulb life has a normal distribution with $\mu = 2000$ hours and $\sigma = 200$ hours. What's the probability that a bulb will last

(a) between 2000 and 2400 hours?

(b) less than 1470 hours?

a) $p(2000 < X < 2400) = p\left(0 < Z < \frac{2400 - 2000}{200}\right)$

b) $p(X < 1470) = p\left(Z < \frac{1470 - 2000}{200}\right)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

$$P(X \leq a) = P\left(\frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}\right) = \Phi\left(\frac{a - \mu}{\sigma}\right).$$

Problem 25.9

If for a certain normal random variable X, $P(X < 500) = 0.5$ and $P(X > 650) = 0.0227$, find the standard deviation of X.

$p(X < 500) = 0.5 \Rightarrow p(Z < \frac{500-\mu}{\sigma}) = 0.5 \Rightarrow \boxed{\frac{500-\mu}{\sigma} = 0}$ (1)

$p(X > 650) = 0.0227 \Rightarrow P(Z < \frac{650-\mu}{\sigma}) 1 - 0.0227 \Rightarrow \boxed{\frac{650-\mu}{\sigma} = 2}$ (2)

from (1) : $\mu = 500$ plug it in (2) : $\frac{150}{\sigma} = 2$

$\Rightarrow \sigma = 75$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

$$P(X \le a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Exponential distribution usually used to model the time between arrivals

Problem 26.7

During busy times buses arrive at a bus stop about every three minutes, so if we measure x in minutes the rate λ of the exponential distribution is $\lambda = \frac{1}{3}$.
(a) What is the probability of having to wait 6 or more minutes for the bus?
(b) What is the probability of waiting between 4 and 7 minutes for a bus?
(c) What is the probability of having to wait at least 9 more minutes for the bus given that you have already waited 3 minutes?

$f(x) = \frac{1}{3} e^{-\frac{1}{3}x} \quad x > 0$

$F(x) = 1 - e^{-\frac{1}{3}x}$

a) $1 - F(6) = e^{-\frac{1}{3} \cdot 6} = 0.135$

b) $F(7) - F(4) = S(4) - S(7) = e^{-4/3} - e^{-7/3} = 0.166$

c) $\underbrace{P(X > 9+3 \mid X > 3) = P(X > 9)}_{\text{memoryless property}} = e^{-9/3} = 0.049$

$f(x) = \lambda e^{-\lambda x} \qquad E(X) = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2} \qquad F(x) = 1 - e^{-\lambda x}$

Problem 26.10 ‡

The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year. What portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year?

we have $p(X<50)=0.3 \Rightarrow 1-e^{-50\lambda}=0.3 \Rightarrow \lambda=\frac{\ln(0.7)}{-50}$

$p(X<80)=1-e^{-80\left(\frac{\ln(0.7)}{-50}\right)}=0.4348$

$f(x)=\lambda e^{-\lambda x}$ $E(X)=\frac{1}{\lambda}$ $Var(X)=\frac{1}{\lambda^2}$ $F(x)=$1- $e^{-\lambda x}$

Problem 26.15 ‡

The time to failure of a component in an electronic device has an exponential distribution with a median of four hours.

Calculate the probability that the component will work without failing for at least five hours.

median : $F(x) = 0.5 \Rightarrow 1-e^{-4\lambda} = 0.5 \Rightarrow \lambda = \frac{\ln(0.5)}{-4}$

$p(X>5) = e^{-5\left(\frac{\ln(0.5)}{-4}\right)}$

$= 0.420$

$f(x) = \lambda e^{-\lambda x}$ $\quad E(X) = \frac{1}{\lambda}$ $\quad Var(X) = \frac{1}{\lambda^2}$ $\quad F(x) = 1- e^{-\lambda x}$

Problem 26.16

Let X be an exponential random variable such that $P(X \leq 2) = 2P(X > 4)$. Find the variance of X.

$$1 - e^{-2\lambda} = 2e^{-4\lambda}$$

$$\Rightarrow 2e^{-4\lambda} + e^{-2\lambda} - 1 = 0$$

$$\Rightarrow \underbrace{2(e^{-2\lambda})^2 + e^{-2\lambda} - 1 = 0}$$

this is a quadratic formula

$$2x^2 + x - 1 = 0 \text{ with } x = e^{-2\lambda}$$

$$\boxed{x = 0.5} \text{ or } x = -1$$

we take the positive

$$\Rightarrow e^{-2\lambda} = 0.5$$

$$\Rightarrow \lambda = \frac{\ln(0.5)}{-2} = 0.346$$

$$Var(X) = \frac{1}{0.346^2} = 8.325$$

$$f(x) = \lambda e^{-\lambda x} \qquad E(X) = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2} \qquad F(x) = 1 - e^{-\lambda x}$$

Problem 27.2
If X has a probability density function given by

$$f(x) = \begin{cases} 4x^2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and the variance.

Gamma distribution is used to model the waiting time until the nth arrivals

First we make sure that the function is a Gamma function

We have $x^{\overset{\alpha-1}{2}} e^{-\overset{\lambda}{2}x}$. Now we check the constant part: $4 \overset{?}{=} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \Rightarrow 4 = \frac{2^3}{2}$ ✓

mean $= \frac{3}{2}$, Variance $= \frac{3}{4}$

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \qquad E(X) = \frac{\alpha}{\lambda} \qquad Var(X) = \frac{\alpha}{\lambda^2}$$

Problem 27.5

Suppose the time (in hours) taken by a professor to grade a complicated exam is a random variable X having a gamma distribution with parameters $\alpha = 3$ and $\lambda = 0.5$. What is the probability that it takes at most 1 hour to grade an exam?

$$f(x) = \frac{0.5e^{-0.5x}(0.5x)^2}{\Gamma(3)}, \quad x > 0$$

integration by parts

$$\begin{array}{lcl} x^2 & & e^{-0.5x} \\ - \;\; 2x & & -2e^{-0.5x} \\ + \;\; 2 & & 4e^{-0.5x} \\ 0 & & -8e^{-0.5x} \end{array}$$

$$= -x^2 2e^{-0.5x} - 8xe^{-0.5x} + (-16)e^{-0.5x} \Big|_0^1$$

$$p(X<1) = \int_0^1 f(x)\,dx = \frac{0.5^3}{\Gamma(3)} \int_0^1 x^2 e^{-0.5x}\,dx$$

$$= \frac{0.5^3}{\Gamma(3)} \left[e^{-0.5x}(-4-8-16) + 16 \right]$$

$$= 0.014$$

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \qquad E(X) = \frac{\alpha}{\lambda} \qquad Var(X) = \frac{\alpha}{\lambda^2}$$

Problem 27.6

Suppose the continuous random variable X has the following pdf:

$$f(x) = \begin{cases} \frac{1}{16}x^2e^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X^3)$.

$$= \frac{1}{16}\underbrace{\int_0^\infty x^{5} e^{-\frac{x}{2}}\, dx}_{\text{we can get rid of this term since we have a complete integral but first we have to complete the Gamma pdf multiplying and dividing by } \lambda^\alpha/\Gamma(\alpha)} = \frac{1}{16}\underbrace{\int_0^\infty \frac{0.5^6}{\Gamma(6)} \cdot \frac{\Gamma(6)}{0.5^6} x^5 e^{-x/2}\, dx}_{=1}$$

$\alpha - 1$ (above the 5), λ (above the $\frac{x}{2}$)

$$= \frac{1}{16} \cdot \frac{5!}{0.5^6}$$

$$= 480$$

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \qquad E(X) = \frac{\alpha}{\lambda} \qquad Var(X) = \frac{\alpha}{\lambda^2}$$