

# Math 209 Midterm 1 (2026) — + Solutions

## Problem 1

Discuss the convergence of the following sequences. Give the limit if it exists, or state “diverges,” with a brief justification.

(a)  $(-1.2)^n$

Let

$$a_n = (-1.2)^n.$$

Since

$$|a_n| = (1.2)^n,$$

and  $1.2 > 1$ , the magnitude grows without bound. Also, the sign changes because of the factor  $(-1)^n$ . Therefore the sequence does not approach any real number.

$$\boxed{(-1.2)^n \text{ diverges.}}$$

(b)  $\frac{\cos^2(n)}{n}$

Let

$$a_n = \frac{\cos^2(n)}{n}.$$

We know that for every  $n$ ,

$$0 \leq \cos^2(n) \leq 1.$$

Since  $n > 0$ , dividing by  $n$  gives

$$0 \leq \frac{\cos^2(n)}{n} \leq \frac{1}{n}.$$

Also,

$$\lim_{n \rightarrow \infty} 0 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

By the Squeeze Theorem,

$$\lim_{n \rightarrow \infty} \frac{\cos^2(n)}{n} = 0.$$

Therefore,

$$\boxed{\lim_{n \rightarrow \infty} \frac{\cos^2(n)}{n} = 0.}$$

(c)  $\frac{4n^3 + n + 1}{2n^3 + 7n^2 + 5}$

Let

$$a_n = \frac{4n^3 + n + 1}{2n^3 + 7n^2 + 5}.$$

Divide the numerator and denominator by  $n^3$ :

$$a_n = \frac{4 + \frac{1}{n^2} + \frac{1}{n^3}}{2 + \frac{7}{n} + \frac{5}{n^3}}.$$

As  $n \rightarrow \infty$ ,

$$\frac{1}{n} \rightarrow 0, \quad \frac{1}{n^2} \rightarrow 0, \quad \frac{1}{n^3} \rightarrow 0.$$

So

$$\lim_{n \rightarrow \infty} a_n = \frac{4}{2} = 2.$$

Therefore,

$$\boxed{\lim_{n \rightarrow \infty} \frac{4n^3 + n + 1}{2n^3 + 7n^2 + 5} = 2.}$$

## Problem 2

Discuss the convergence of each series.

(a)  $\sum_{n=1}^{\infty} \frac{n}{4^n}$

Let

$$a_n = \frac{n}{4^n}.$$

Use the Ratio Test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/4^{n+1}}{n/4^n} = \frac{n+1}{4n}.$$

Now take the limit:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n+1}{4n} = \frac{1}{4}.$$

Since

$$\frac{1}{4} < 1,$$

the series converges by the Ratio Test.

$$\boxed{\sum_{n=1}^{\infty} \frac{n}{4^n} \text{ converges.}}$$

(b)  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

We use the Integral Test with

$$f(x) = \frac{1}{x(\ln x)^2}, \quad x \geq 2.$$

This function is positive and continuous for  $x \geq 2$ , and decreasing for large  $x$ . So the Integral Test applies.

Compute:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx.$$

Let

$$u = \ln x, \quad du = \frac{dx}{x}.$$

Then

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^{\infty} \frac{1}{u^2} du = \left[ -\frac{1}{u} \right]_{\ln 2}^{\infty} = \frac{1}{\ln 2}.$$

This is finite, so the integral converges. Therefore the series also converges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \text{ converges.}$$

(c)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This is an alternating series with

$$b_n = \frac{1}{\sqrt{n}}.$$

We check the conditions of the Alternating Series Test:

$$b_n > 0, \quad b_{n+1} < b_n, \quad \lim_{n \rightarrow \infty} b_n = 0.$$

So the series converges.

Now check absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a  $p$ -series with

$$p = \frac{1}{2} < 1,$$

so it diverges.

Therefore the series converges, but not absolutely.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ converges conditionally.}$$

(d)  $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+1}\right)^n$

Let

$$a_n = \left(\frac{n+1}{3n+1}\right)^n.$$

Use the Root Test:

$$\sqrt[n]{a_n} = \frac{n+1}{3n+1}.$$

Take the limit:

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{n+1}{3n+1} = \frac{1}{3}.$$

Since

$$\frac{1}{3} < 1,$$

the series converges by the Root Test.

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+1}\right)^n \text{ converges.}$$

### Problem 3

Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n+1}} (x+1)^n.$$

State clearly which endpoint(s), if any, are included in the interval of convergence.

### Radius of convergence

Let

$$a_n = \frac{3^n}{\sqrt{n+1}} (x+1)^n.$$

Use the Root Test:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{3^n}{\sqrt{n+1}}} |x+1| = 3 \cdot \frac{|x+1|}{(n+1)^{1/(2n)}}.$$

Since

$$(n+1)^{1/(2n)} \rightarrow 1,$$

we get

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 3|x+1|.$$

The series converges when

$$3|x + 1| < 1.$$

So

$$|x + 1| < \frac{1}{3}.$$

Hence the radius of convergence is

$$\boxed{R = \frac{1}{3}}.$$

## Interval of convergence

Since the center is  $-1$ , we have

$$-1 - \frac{1}{3} < x < -1 + \frac{1}{3}.$$

So the open interval is

$$\left(-\frac{4}{3}, -\frac{2}{3}\right).$$

## Check the endpoints

**Left endpoint:**  $x = -\frac{4}{3}$

Then

$$x + 1 = -\frac{1}{3},$$

and the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n+1}} \left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

This is an alternating series. The terms

$$\frac{1}{\sqrt{n+1}}$$

decrease to 0, so the series converges by the Alternating Series Test.

Therefore, the left endpoint is included.

**Right endpoint:**  $x = -\frac{2}{3}$

Then

$$x + 1 = \frac{1}{3},$$

and the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n+1}} \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}.$$

This is a  $p$ -series with

$$p = \frac{1}{2} < 1,$$

so it diverges.

Therefore, the right endpoint is not included.

## Final answer

The radius of convergence is

$$R = \frac{1}{3}.$$

The interval of convergence is

$$\left[-\frac{4}{3}, -\frac{2}{3}\right).$$