Power Series

Power Series

A **power series** centered at $x_0 \in \mathbb{R}$ is

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where the coefficients a_n are real numbers.

If $x_0 = 0$, the series becomes

$$\sum_{n=0}^{\infty} a_n x^n,$$

and is called a power series centred **at the origin**.

Radius of Convergence

For the series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

there exists a number $R \in [0, \infty]$, called the **radius of convergence**, such that

$$\begin{aligned} |x - x_0| < R &\implies \text{the series converges absolutely,} \\ |x - x_0| > R &\implies \text{the series diverges.} \end{aligned}$$

If $R = \infty$, the series converges absolutely for every real number x.

How to Compute the Radius R

Use the **Ratio Test** or the **Root Test**.

If one of the limits exists

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|, \qquad \text{or} \qquad L = \lim_{n \to \infty} \sqrt[n]{|a_n|},$$

then

$$R = \begin{cases} \dfrac{1}{L}, & 0 < L < \infty, \\ \infty, & L = 0, \\ 0, & L = \infty. \end{cases}$$

Interval of Convergence

- If $R = \infty$, the series converges for all $x \in \mathbb{R}$. The interval of convergence is $\mathbb{R}$.
- If $R < \infty$, the series converges absolutely for

$$x \in (x_0 - R,\ x_0 + R).$$

- At the boundary points $x = x_0 - R$ and $x = x_0 + R$, the rule above gives no information. Each endpoint must be tested separately. The series may converge conditionally or diverge.

Example: Geometric Series

Consider
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .$$
Here $x_0 = 0$ and $a_n = 1$.

Using the Ratio Test,
$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|, \quad (x \neq 0).$$

$$|x| < 1 \implies \text{convergence},$$
$$|x| > 1 \implies \text{divergence},$$
$$|x| = 1 \implies \text{test inconclusive}.$$

Therefore, the radius of convergence is
$$R = 1.$$

Hence, the series converges absolutely for
$$-1 < x < 1.$$

We now study the boundary points.

- If $x = 1$, the series becomes
$$\sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots ,$$
which clearly diverges.

- If $x = -1$, the series becomes
$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots ,$$
whose partial sums oscillate between 1 and 0. Hence, the series does not converge.

The interval of convergence is therefore
$$(-1, 1).$$
and on this whole interval the convergence is **absolute**.

Consider the power series

$$\sum_{n=1}^{\infty} \frac{3^n(x-3)^n}{4^n n}.$$

This is a power series centered at

$$x_0 = 3.$$

Apply the Ratio Test to the general term:

$$\left| \frac{3^{n+1}(x-3)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n n}{3^n(x-3)^n} \right| = \left| \frac{3}{4} \cdot \frac{n}{n+1} \cdot (x-3) \right|.$$

Taking the limit as $n \to \infty$,

$$\frac{3}{4}|x-3|.$$

The series converges when

$$\frac{3}{4}|x-3| < 1, \qquad \text{that is} \qquad |x-3| < \frac{4}{3}.$$

Hence, the radius of convergence is

$$R = \frac{4}{3}.$$

The inequality

$$|x-3| < \frac{4}{3}$$

means that x is at a distance less than $\frac{4}{3}$ from 3. This is equivalent to the open interval

$$3 - \frac{4}{3} < x < 3 + \frac{4}{3},$$

that is

$$\frac{5}{3} < x < \frac{13}{3}.$$

- At $x = \dfrac{13}{3}$, the series becomes

  $$\sum_{n=1}^{\infty} \frac{1}{n},$$

  which diverges.

- At $x = \dfrac{5}{3}$, the series becomes

  $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

  which converges conditionally.

Interval of convergence : $\left[\frac{5}{3}, \frac{13}{3}\right)$.

Consider

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Here $a_n = \frac{1}{n!}$.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0.$$

The limit is 0 for every real x.

Conclusion

$$R = \infty, \qquad \text{interval } = (-\infty, \infty).$$

This series defines

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example 4

Consider the power series

$$\sum_{n=2}^{\infty} \frac{x^n}{n \log n},$$

We evaluate

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n \log n}\right)^{1/n}.$$

Taking the natural logarithm,

$$\ln\left(\left(\frac{1}{n \log n}\right)^{1/n}\right) = -\frac{\ln(n \log n)}{n}.$$

$$\lim_{n\to\infty} \frac{\ln(n \log n)}{n} = 0$$

(by L'Hopital's rule). Hence,

$$\lim_{n\to\infty} \left(\frac{1}{n \log n}\right)^{1/n} = e^0 = 1.$$

Therefore,

$$\lim_{n\to\infty} |a_n x^n|^{1/n} = |x|.$$

This gives

$$|x| < 1 \Rightarrow \text{convergence}, \qquad |x| > 1 \Rightarrow \text{divergence}.$$

Thus, the radius of convergence is $R = 1$. We now study the boundary points.

For $x = 1$, the series becomes

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}.$$

Using the Integral Test with $f(x) = \frac{1}{x \log x}$,

$$\int_2^{\infty} \frac{1}{x \log x}\, dx = \int_{\log 2}^{\infty} \frac{1}{u}\, du = \infty,$$

so the series diverges.

For $x = -1$, the series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}.$$

This is an alternating series where $\frac{1}{n \log n}$ is positive, decreasing (for $n \geq 3$), and tends to 0. Hence, it converges conditionally by the Alternating Series Test.

Conclusion

$$\sum_{n=2}^{\infty} \frac{x^n}{n \log n} \text{ converges for } x \in [-1, 1).$$

Radius of convergence: $R = 1$

Interval of convergence: $[-1, 1)$

Main Properties of Power Series

Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

be a power series with radius of convergence $R > 0$.

For every x such that $|x - x_0| < R$, the series converges. Therefore, the series **defines a function**

$$f : (x_0 - R,\ x_0 + R) \longrightarrow \mathbb{R}.$$

Differentiation Term by Term

Start with the power series written as an infinite polynomial:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots . \end{aligned}$$

We differentiate each term exactly as we do for polynomials.

$$\begin{aligned} \frac{d}{dx}\Big[a_0(x - x_0)^0\Big] &= \frac{d}{dx}[a_0] = 0, \\ \frac{d}{dx}\Big[a_1(x - x_0)\Big] &= a_1, \\ \frac{d}{dx}\Big[a_2(x - x_0)^2\Big] &= 2a_2(x - x_0), \\ \frac{d}{dx}\Big[a_3(x - x_0)^3\Big] &= 3a_3(x - x_0)^2, \end{aligned}$$

and so on.

In general,

$$\frac{d}{dx}\Big[a_n(x - x_0)^n\Big] = n\, a_n (x - x_0)^{n-1}.$$

Therefore, the derivative becomes

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots \\ &= \sum_{n=1}^{\infty} n\, a_n (x - x_0)^{n-1}. \end{aligned}$$

Reindexing to start again from $n = 0$

Let $k = n - 1$. Then $n = k + 1$. The series can be written as

$$f'(x) = \sum_{k=0}^{\infty} (k + 1) a_{k+1} (x - x_0)^k.$$

The important fact is that this new series has the **same radius of convergence** R.

We know that for $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

This means the power series defines the function

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Differentiate both sides.

Differentiate the series term by term

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

Differentiate the function

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

Therefore,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots , \qquad |x| < 1.$$

Higher–Order Differentiation

We know that for $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

First derivative

Differentiating term by term,

$$\sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}.$$

Second derivative

Differentiate again term by term:

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = 2 + 6x + 12x^2 + 20x^3 + \cdots = \frac{2}{(1-x)^3}.$$

Third derivative

$$\sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3} = \frac{6}{(1-x)^4}.$$

General formula

After differentiating k times,

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, x^{n-k} = \frac{k!}{(1-x)^{k+1}}, \qquad |x| < 1.$$

Integration Term by Term

Start from

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Inside the interval of convergence, the series behaves like a polynomial, so we integrate each term as usual.

$$\int a_n (x - x_0)^n \, dx = \frac{a_n}{n+1}(x - x_0)^{n+1} + C.$$

Hence, the **indefinite integral** is

$$\int f(x) \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}.$$

Let $k = n + 1$. Then $n = k - 1$, and

$$\int f(x) \, dx = C + \sum_{k=1}^{\infty} \frac{a_{k-1}}{k}(x - x_0)^k.$$

Definite integral

For x inside the interval of convergence,

$$\int_{x_0}^{x} f(t) \, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}.$$

(Here the constant C disappears.)

This new series has the **same radius of convergence** R.

For $|x| < 1$, we know

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}.$$

Integrate both sides from 0 to x

$$\int_0^x \sum_{n=0}^{\infty} t^n \, dt = \int_0^x \frac{1}{1-t} \, dt.$$

Integrate the series term by term

$$\sum_{n=0}^{\infty} \int_0^x t^n \, dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

Integrate the function

$$\int_0^x \frac{1}{1-t} \, dt = \Big[-\ln(1-t) \Big]_0^x = -\ln(1-x).$$

Final result

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \qquad |x| < 1.$$

Rewriting the index,

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

How a Function Becomes a Power Series

Assume a function f is given by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with radius of convergence $R > 0$.

Inside $(x_0 - R,\ x_0 + R)$, the series behaves like a polynomial. So we can differentiate term by term.

Differentiate once:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}.$$

Evaluate at $x = x_0$. All terms with $(x - x_0)$ vanish except the first one:

$$f'(x_0) = a_1.$$

Differentiate again:

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}.$$

Evaluate at $x = x_0$:

$$f''(x_0) = 2!\, a_2.$$

Continuing this process, we obtain

$$f^{(n)}(x_0) = n!\, a_n.$$

Hence, the coefficients of the power series are

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Taylor Series

Substituting the coefficients back into the series gives

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \qquad |x - x_0| < R.$$

This is called the **Taylor series of f at** x_0.

Maclaurin Series

If $x_0 = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

called the **Maclaurin series**.

Power Series of $f(x) = e^x$

We want to express e^x as a power series centered at 0 (Maclaurin series).

First compute the derivatives of $f(x) = e^x$:

$$f(x) = e^x,$$
$$f'(x) = e^x,$$
$$f''(x) = e^x,$$
$$f^{(3)}(x) = e^x, \quad \dots$$

All derivatives are equal to e^x.

Now evaluate these derivatives at $x = 0$:

$$f(0) = e^0 = 1,$$
$$f'(0) = e^0 = 1,$$
$$f''(0) = e^0 = 1,$$
$$f^{(3)}(0) = e^0 = 1, \quad \dots$$

So for every n,

$$f^{(n)}(0) = 1.$$

Recall the Maclaurin formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Substitute $f^{(n)}(0) = 1$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Writing the first terms:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

This power series converges for every real number x (its radius of convergence is ∞).

Note that t, if $f(x) = \sum a_n x^n,\ g(x) = \sum b_n x^n$, then

$$(f(x) + g(x) = \sum (a_n + b_n) x^n.$$

Hyperbolic Functions from the Power Series of e^x

Recall the Maclaurin series of the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots .$$

The hyperbolic functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Computation of $\sinh x$

Subtract the two series:

$$e^x - e^{-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right).$$

Now subtract term by term:

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots .$$

Divide by 2:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots .$$

Hence,

$$\boxed{\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}} \qquad (R = \infty).$$

Computation of $\cosh x$

Add the two series:

$$e^x + e^{-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right).$$

Add term by term:

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots .$$

Divide by 2:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

Hence,

$$\boxed{\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}} \qquad (R = \infty).$$

Representation of Functions by Power Series

A function f is represented by a power series at x_0 if

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{for } |x - x_0| < R.$$

Method 1: Maclaurin Series (Taylor at 0)

If f is smooth near 0, we compute derivatives at 0:

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

So,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}.$$

Example: $f(x) = e^x$

Since $f^{(n)}(0) = 1$,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Method 2: Substitution

Start from

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (|x|<1).}$$

If a function can be written as

$$\frac{1}{1-g(x)},$$

then substitute $x = g(x)$:

$$\frac{1}{1-g(x)} = \sum_{n=0}^{\infty} \big(g(x)\big)^n \quad |g(x)| < 1.$$

Example 1: $\dfrac{1}{2+3x}$

$$\frac{1}{2+3x} = \frac{1}{2} \cdot \frac{1}{1-\left(-\frac{3}{2}x\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{2^{n+1}} x^n, \quad |x| < \frac{2}{3}.$$

Example 2: $\dfrac{1}{1+x^2}$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1.$$

Method 3: Differentiation and Integration

If

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then

Differentiate:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Integrate from 0 to x:

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}.$$

These operations keep the same radius of convergence.

This is how we obtain series for functions such as $\ln(1+x)$ and $\arctan x$ starting from the geometric series.

Important Power Series to Know

Function	Power Series (Maclaurin)	Interval of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$\lvert x \rvert < 1$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$-1 < x \leq 1$
$\arctan x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$\lvert x \rvert \leq 1$

Exercises

1) Show that

$$\lim_{x\to 0} \frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3} = \frac{1}{3}.$$

Solution. For $|x| < 1$,

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

Subtract $x - \frac{x^2}{2}$:

$$\ln(1+x) - x + \frac{x^2}{2} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - x + \frac{x^2}{2} = \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

Divide by x^3:

$$\frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3} = \frac{1}{3} - \frac{x}{4} + \cdots .$$

Let $x \to 0$:

$$\boxed{\lim_{x\to 0} \frac{\ln(1+x) - x + \frac{x^2}{2}}{x^3} = \frac{1}{3}.}$$

2) Show that

$$\lim_{x\to 0}\frac{1-\cos x-\frac{x^2}{2}}{x^4}=\frac{1}{24}.$$

Solution. For all real x,

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots.$$

Compute the numerator:

$$\begin{aligned}1-\cos x-\frac{x^2}{2}&=1-\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)-\frac{x^2}{2}\\&=-\frac{x^2}{2}+\frac{x^2}{2}-\frac{x^4}{24}+\frac{x^6}{720}-\cdots\\&=-\frac{x^4}{24}+\frac{x^6}{720}-\cdots.\end{aligned}$$

Divide by x^4:

$$\frac{1-\cos x-\frac{x^2}{2}}{x^4}=-\frac{1}{24}+\frac{x^2}{720}-\cdots.$$

Let $x\to 0$:

$$\boxed{\lim_{x\to 0}\frac{1-\cos x-\frac{x^2}{2}}{x^4}=-\frac{1}{24}.}$$

3) Evaluate

$$\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{2}\right)^n.$$

Solution. For $|x|<1$,

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n.$$

Substitute $x=\frac{1}{2}$:

$$\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{2}\right)^n=\frac{1}{\left(1-\frac{1}{2}\right)^2}=\frac{1}{\left(\frac{1}{2}\right)^2}=\boxed{4}.$$

4) Show that

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}, \qquad |x| < 1,$$

and compute

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

Solution.

Starting from the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

differentiate twice and multiply appropriately by x. This yields

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}, \qquad |x| < 1.$$

Substituting $x = \dfrac{1}{3}$,

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{\frac{1}{3}\left(1+\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)^3} = \frac{4}{9} \cdot \frac{27}{8} = \boxed{\frac{3}{2}}.$$