

Double Integrals in Polar Coordinates

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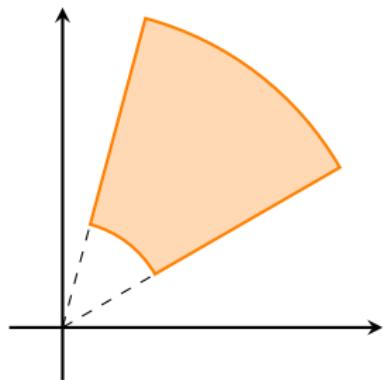
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March 25, 2024

1 Double Integrals in Polar Coordinates

Polar coordinates are defined by $x = r \cos \theta$, $y = r \sin \theta$. The area of the shaded region

$$R = \{(r, \theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$



The integral of a continuous function $f(x, y)$ over a polar rectangle R given by $a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, is

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

Example

- ① Find $\iint_R (2x - y) dA$ if R is the region in the first quadrant bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.
- ② Find $\iint_R e^{-x^2-y^2} dA$ if D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y -axis.

Theorem

If f is continuous over a polar region of the form

$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

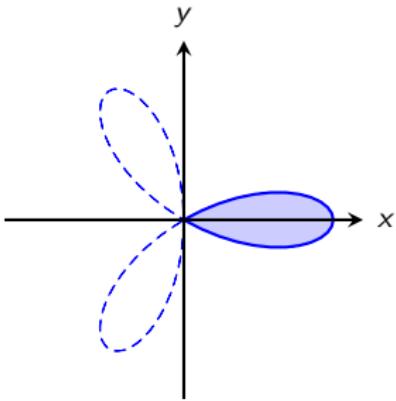
then

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Example

The area of one loop of the rose $r = \cos(3\theta)$

$$A = \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r dr d\theta = \frac{\pi}{2}.$$

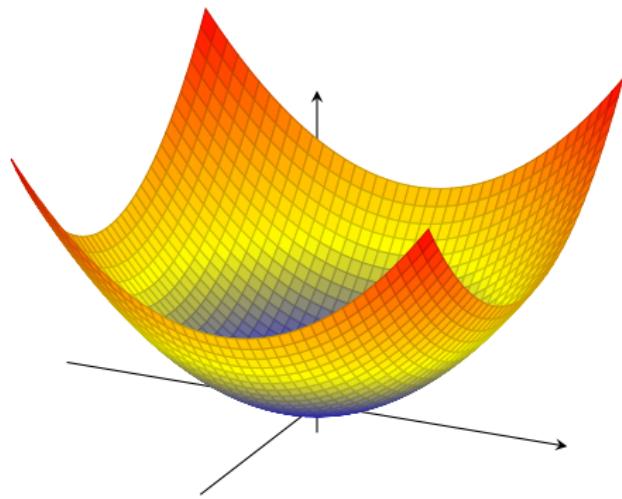


Example

The volume under the paraboloid $z = f(x, y) = x^2 + y^2$ and above the disc $x^2 + y^2 < 3$.

In polar coordinates, the disc is parameterized by: $x = r \cos \theta$, $y = r \sin \theta$, with $r \in [0, 3]$ and $\theta \in [0, 2\pi]$. Since in polar coordinates $f(x, y)$ is transformed to $g(r, \theta) = r^2$. Hence the volume under the paraboloid $z = x^2 + y^2$ and above the disc $x^2 + y^2 < 3$ is

$$V = \int_0^3 \int_0^{2\pi} r^2 dr d\theta = 18\pi.$$



Example

Evaluation of the following integral by converting to polar coordinates

$$\int_0^a \int_0^{\sqrt{a^2 - y^2}} y dx dy$$

$$\begin{aligned} \int_0^a \int_0^{\sqrt{a^2 - y^2}} y dx dy &= \int_0^a \int_0^{\frac{\pi}{2}} r^2 \sin \theta dr d\theta \\ &= \frac{a^3}{3}. \end{aligned}$$

Example

Evaluation of the following integral by converting to polar

$$\text{coordinates } \int_1^2 \int_0^x \frac{1}{x^2 + y^2} dy dx$$

$$\int_1^2 \int_0^x \frac{1}{x^2 + y^2} dy dx =$$

Example

Evaluation of the following integral by converting to polar

coordinates $\int_0^2 \int_0^{\sqrt{4-y^2}} \cos(x^2 + y^2) dx dy$

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{4-y^2}} \cos(x^2 + y^2) dx dy &= \int_0^2 \int_0^{\frac{\pi}{2}} r \cos(r^2) dr d\theta \\ &= \frac{\pi}{4} \sin 4. \end{aligned}$$

Example

Evaluation of the following integral by converting to polar

$$\text{coordinates } \int_0^2 \int_0^{\sqrt{2x-x^2}} xy dy dx$$

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{2x-x^2}} xy dy dx &= \int_0^2 \int_0^{\sqrt{4-(x-1)^2}} r dr d\theta \\ &= . \end{aligned}$$

Example

Evaluation of the following integral by converting to polar coordinates

$$\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{\frac{3}{2}} dy dx = \int_0^3 \int_0^{\frac{\pi}{2}} r^4 dr d\theta = \frac{3^5 \pi}{10};$$

Example

Evaluation of the following integral by converting to polar

$$\text{coordinates } \int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} (x^2 + y^2)^{\frac{3}{2}} dy dx \int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^4 dr d\theta = \frac{a^5 \pi}{5};$$

Example

Evaluation of the following integral by converting to polar

coordinates $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) \tan^{-1}(y/x) dx dy =$

$$\int_0^a \int_0^{\frac{\pi}{2}} r^2 \theta r dr d\theta = \frac{\pi^2 a^4}{32}.$$