104 PHYS

Ch. 24

Gauss's Law

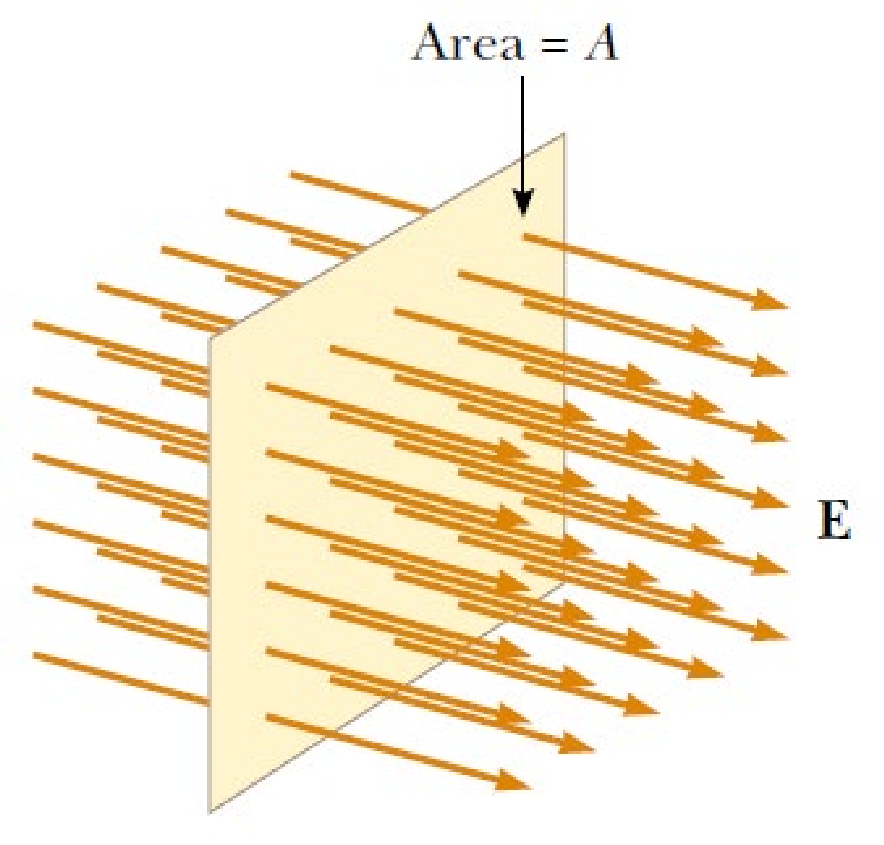
Dr. Ayman Alismail ©

Contents

Electric Flux

- The electric flux Φ_E is proportional to the number of electric field lines penetrating some surface.
- The electric flux Φ_E is the product of the magnitude of the electric field E and surface area A perpendicular to the field.
- The electric flux Φ_E has units of newton-meters squared per coulomb $(\mathrm{N \cdot m^2/C})$.
- If the surface under consideration is perpendicular to the field:

$$\Phi_E = EA$$



Field lines representing a uniform electric field penetrating a plane of area A perpendicular to the field. The electric flux Φ_E through this area is equal to EA.

Electric Flux

Example 24.01

What is the electric flux through a sphere that has a radius of 1.00 m and carries a charge of +1.00 μC at its center?

The magnitude of the electric field 1.00 m from this charge is:

$$E = k_e \frac{q}{r^2}$$

$$E = 9 \times 10^9 \times \frac{1.00 \times 10^{-6}}{(1.00)^2}$$

$$E = 9 \times 10^3 \ \mathrm{N/C}$$

The field points radially outward and is therefore everywhere perpendicular to the surface of the sphere. The flux through the sphere is thus:

$$\Phi_E = EA$$

$$\Phi_E = E(4\pi r^2)$$

$$\Phi_E = 9 \times 10^3 \times 4 \times 3.14 \times (1)^2$$

$$\Phi_E = 1.13 \times 10^5 \ \mathrm{N \cdot m^2/C}$$

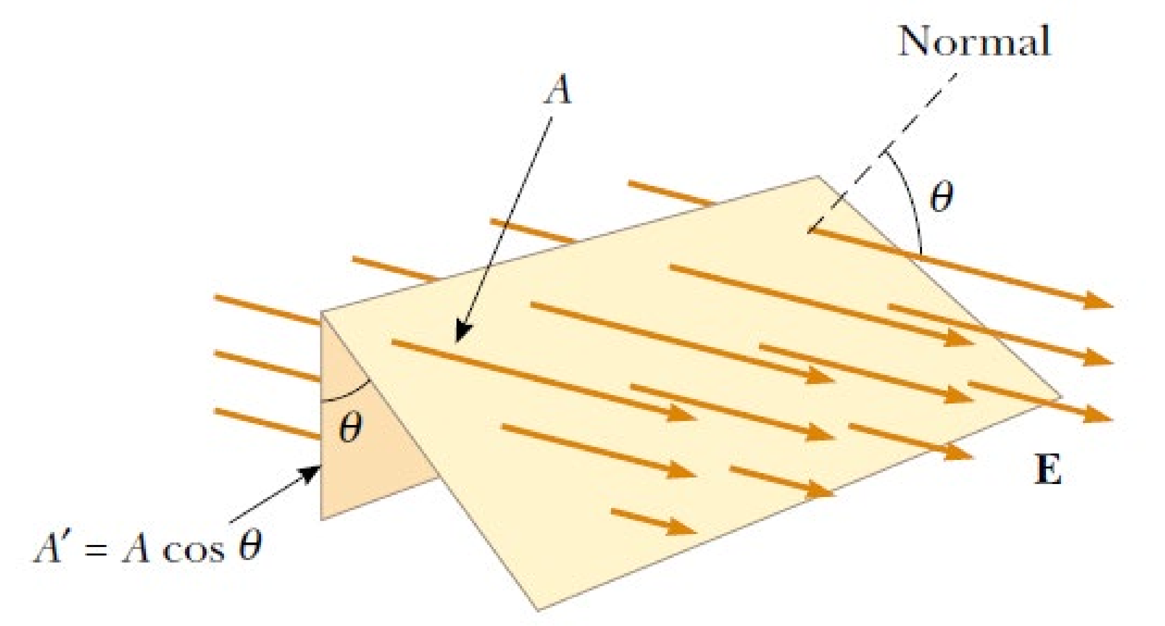
Electric Flux

- If the surface under consideration is not perpendicular to the field:

$$\Phi_{E,\,A'} = EA' \cos(180)$$

$$\Phi_{E,A} = EA \cos\theta$$

- The electric flux Φ_E has a maximum value EA when $\theta = 0°$
- The electric flux Φ_E is zero when $\theta = 90°$



Field lines representing a uniform electric field penetrating an area A that is at an angle θ to the field. Because the number of lines that go through the area A' is the same as the number that go through A, the flux through A' is equal to the flux through A and is given by $\Phi_E = EA\cos\theta$.

Electric Flux

Problem 24.04

Consider a closed triangular box resting within a horizontal electric field of magnitude $E = 7.80 \times 10^4$ N/C as shown in the following figure. Calculate the electric flux through (a) the vertical rectangular surface, (b) the slanted surface, and (c) the entire surface of the box.

(a) $A' = 0.10 \times 0.30 = 0.03 \text{ m}^2$

$\Phi_{E,A'} = EA' \cos\theta$

$\Phi_{E,A'} = 7.80 \times 10^4 \times 0.03 \times \cos(180)$

$\Phi_{E,A'} = -2.34 \times 10^3 \text{ N} \cdot \text{m}^2/\text{C}$

(b) $A = 0.30 \times \dfrac{0.1}{\cos(60)} = 0.06 \text{ m}^2$

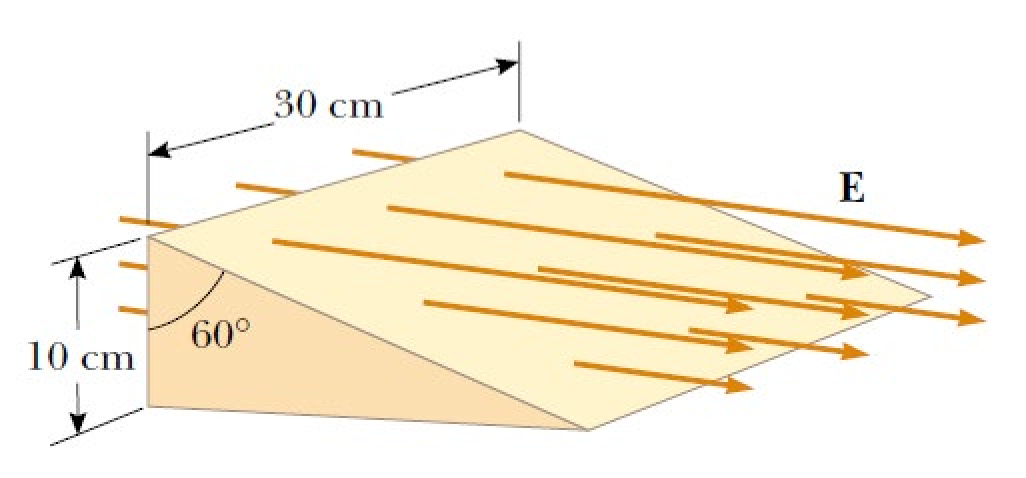
$\Phi_{E,A} = EA \cos\theta$

$\Phi_{E,A} = 7.80 \times 10^4 \times 0.06 \times \cos(60)$

$\Phi_{E,A} = 2.34 \times 10^3 \text{ N} \cdot \text{m}^2/\text{C}$

(c) $\Phi_{E,total} = -2.34 \times 10^3 + 2.34 \times 10^3 + 0 + 0 + 0$

$\Phi_{E,total} = 0$



Electric Flux

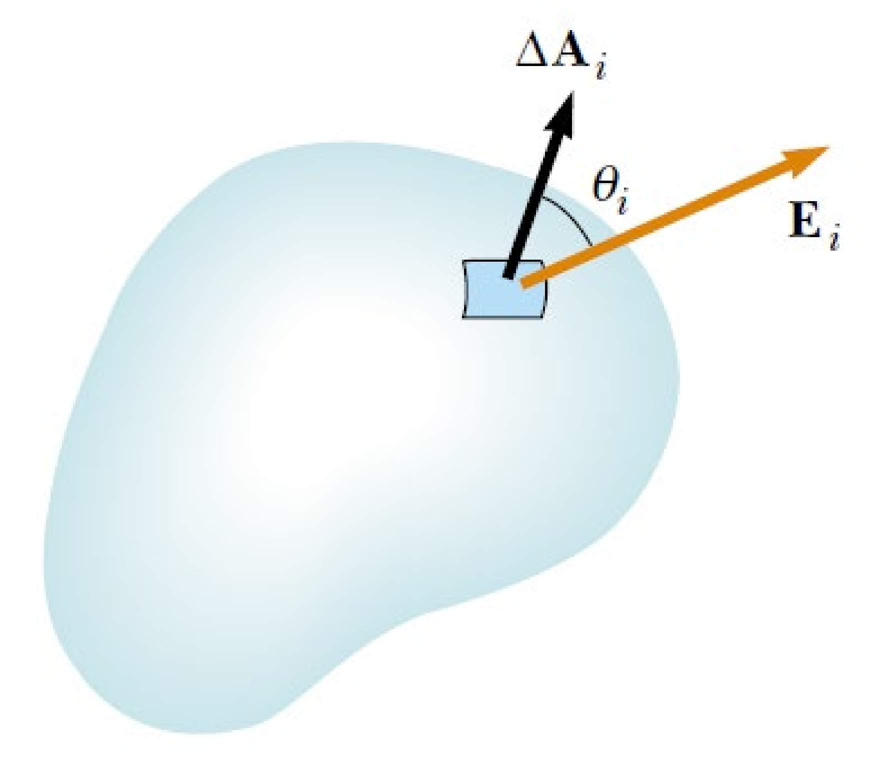
King Saud University

- In more general situations, the electric field may vary over a surface.
- The variation in the electric field over one element can be neglected if the element is sufficiently small.

$$\Delta\Phi_E = E_i \Delta A_i \cos\theta_i = \vec{E}_i \cdot \Delta\vec{A}_i$$

- If we let the area of each element approach zero, then the number of elements approaches infinity and the sum is replaced by an integral. Therefore, the general definition of electric flux is:

$$\Phi_E = \lim_{\Delta A_i \to 0} \sum \vec{E}_i \cdot \Delta\vec{A}_i = \int_{\text{surface}} \vec{E} \cdot d\vec{A}$$

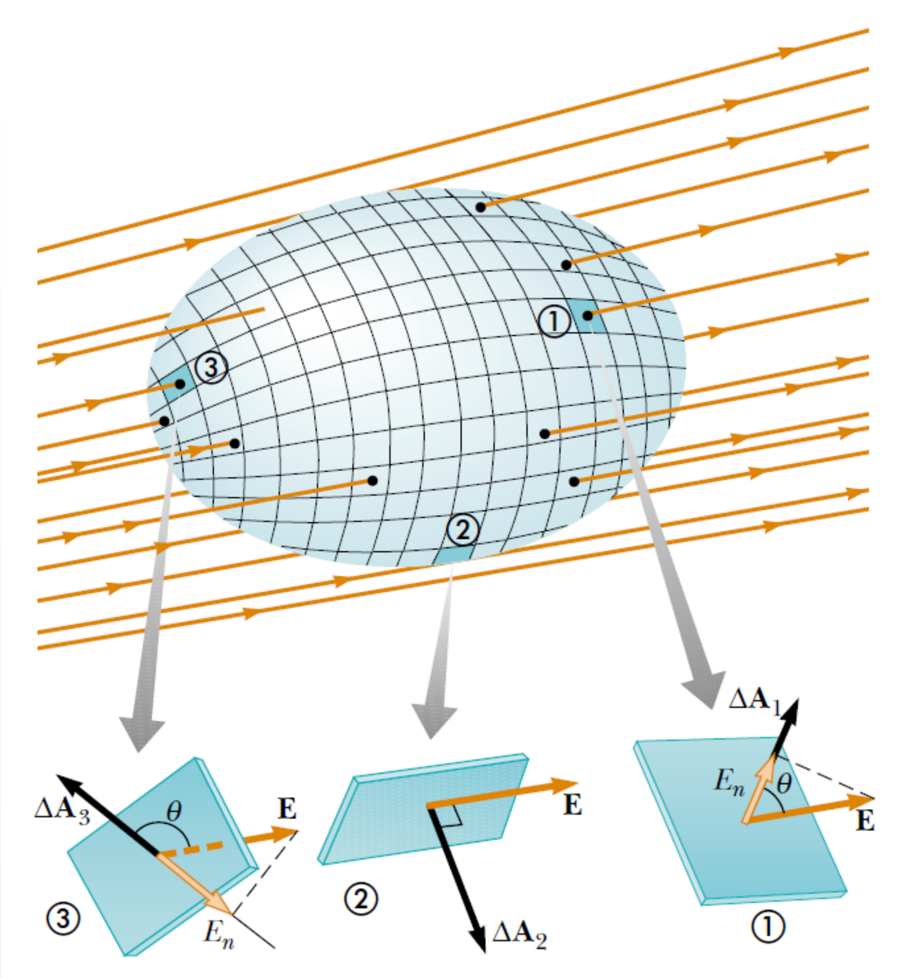


A small element of surface area ΔA_i. The electric field makes an angle θ_i with the vector ΔA_i, defined as being normal to the surface element, and the flux through the element is equal to $E_i \Delta A_i \cos\theta_i$.

Electric Flux

- A closed surface is defined as one that divides space into an inside and an outside region, so that one cannot move from one region to the other without crossing the surface.
- The net flux through the surface is proportional to the net number of lines leaving the surface, where the net number means the number leaving the surface minus the number entering the surface.
- If more lines are leaving than entering, the net flux is positive. If more lines are entering than leaving, the net flux is negative.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E_n \, dA$$



A closed surface in an electric field. The area vectors ΔA_i are, by convention, normal to the surface and point outward. The flux through an area element can be positive (element ①), zero (element ②), or negative (element ③).

Electric Flux

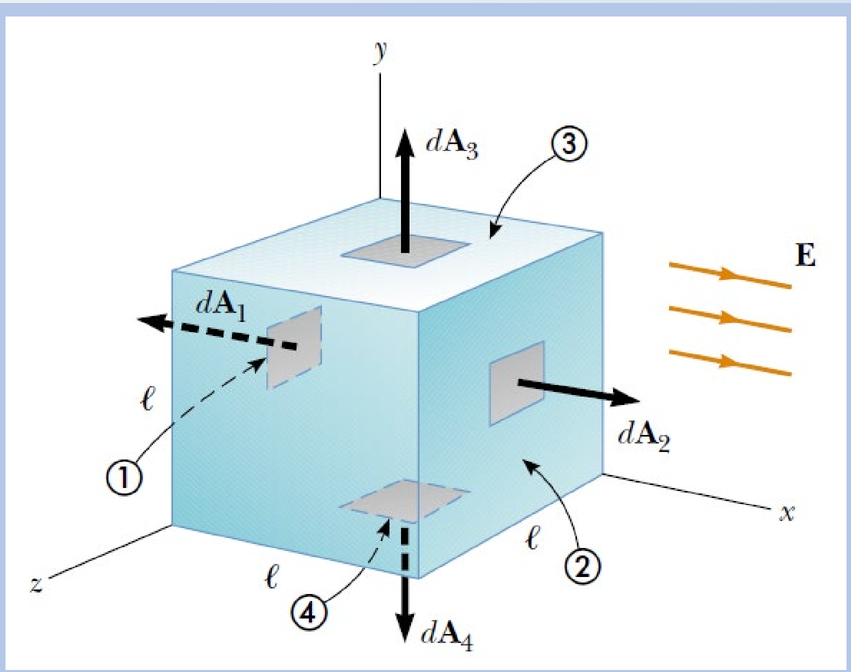
Example 24.02

Consider a uniform electric field $\vec{E}$ oriented in the x direction. Find the net electric flux through the surface of a cube of edge length l, oriented as shown in the following figure.

The net flux is the sum of the fluxes through all faces of the cube. First, note that the flux through four of the faces (③, ④, and the unnumbered ones) is zero because $\vec{E}$ is perpendicular to $d\vec{A}$ on these faces.

The net flux through faces ① and ② is:

$$\Phi_E = \int_1 \vec{E} \cdot d\vec{A} + \int_2 \vec{E} \cdot d\vec{A}$$



A closed surface in the shape of a cube in a uniform electric field oriented parallel to the x axis. Side ④ is the bottom of the cube, and side ① is opposite side ②.

Electric Flux

Example 24.02

Consider a uniform electric field $\vec{E}$ oriented in the x direction. Find the net electric flux through the surface of a cube of edge length l, oriented as shown in the following figure.

For face ①, $\vec{E}$ is constant and directed inward but $d\vec{A}_1$ is directed outward ($\theta = 180°$); thus, the flux through this face is:

$$\int_1 \vec{E}\cdot d\vec{A} = \int_1 E\left(\cos(180)\right) dA = -E\int_1 dA = -EA = -El^2$$

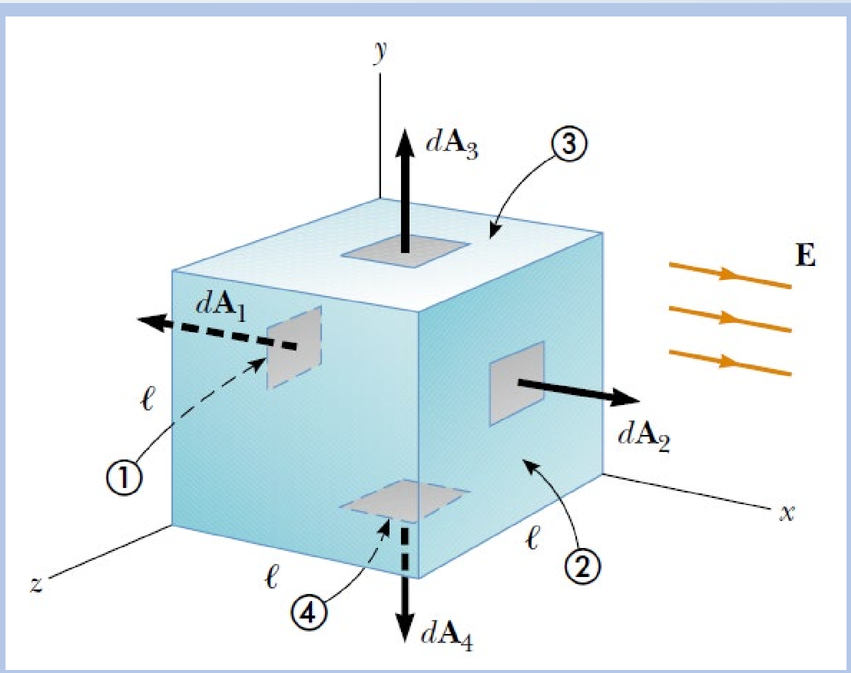
For face ②, $\vec{E}$ is constant and outward and in the same direction as $d\vec{A}_2$ is directed outward ($\theta = 0°$); hence, the flux through this face is:

$$\int_2 \vec{E}\cdot d\vec{A} = \int_2 E\left(\cos(0)\right) dA = E\int_2 dA = EA = El^2$$

Therefore, the net flux over all six faces is:

$$\Phi_E = -El^2 + El^2 + 0 + 0 + 0 + 0$$

$$\Phi_E = 0$$



A closed surface in the shape of a cube in a uniform electric field oriented parallel to the x axis. Side ④ is the bottom of the cube, and side ① is opposite side ②.

Gauss's Law

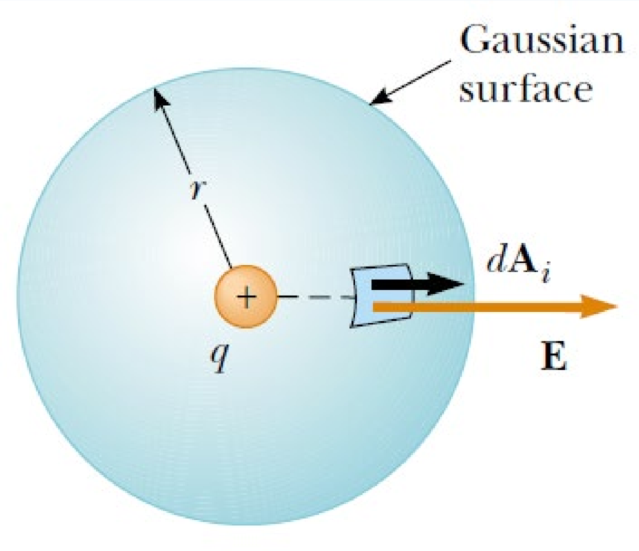
- Gauss's law describes the relationship between the net electric flux through a closed surface (often called a gaussian surface) and the charge enclosed by the surface.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E\, dA = E \oint dA$$

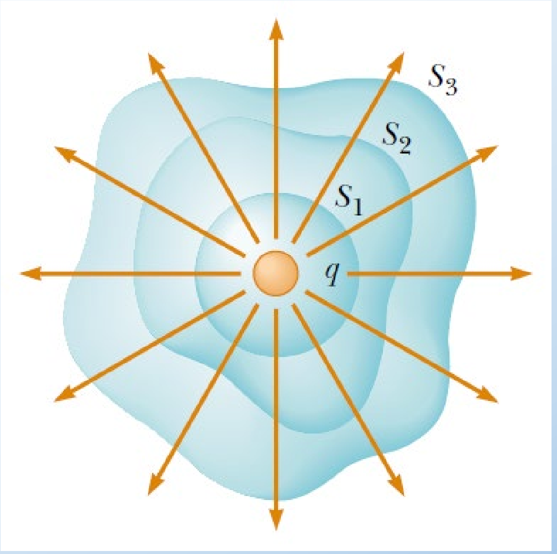
$$\Phi_E = \frac{k_e q}{r^2}(4\pi r^2) = 4\pi k_e q$$

$$\Phi_E = \frac{q}{\epsilon_0}$$

- The net flux through any closed surface surrounding a point charge q is given by q/ϵ_0 and is independent of the shape of that surface.



A spherical gaussian surface of radius r surrounding a point charge q. When the charge is at the center of the sphere, the electric field is everywhere normal to the surface and constant in magnitude.



Closed surfaces of various shapes surrounding a charge q. The net electric flux is the same through all surfaces.

Gauss's Law

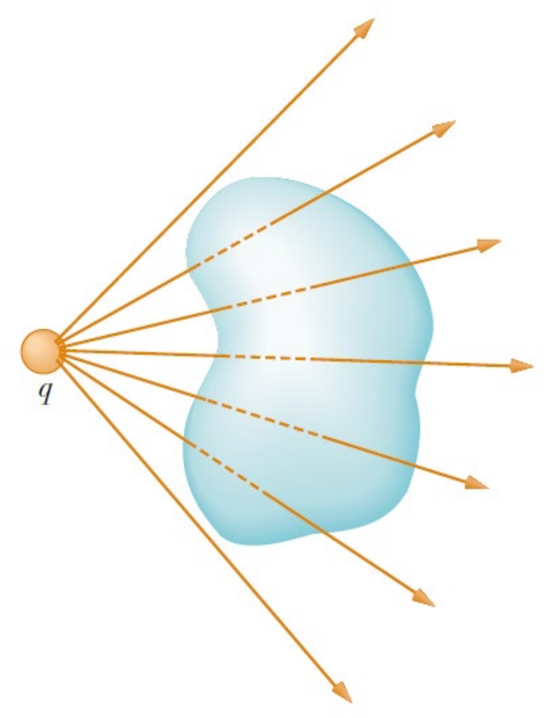
- When the point charge located <u>outside</u> a closed surface of arbitrary shape, the net electric flux through a closed surface that surrounds no charge is zero.

- In the case of many point charges (or a continuous distribution of charge), the electric flux due to many charges is the vector sum of the electric fields produced by the individual charges:

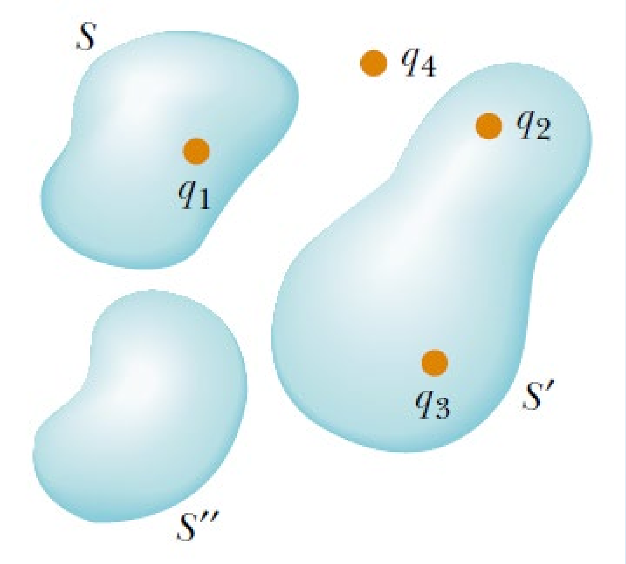
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint \left(\vec{E}_1 + \vec{E}_2 + \cdots\right) \cdot d\vec{A}$$

- Gauss's law, which is a generalization of what we have just described, states that the net flux through any closed surface is:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}$$



A point charge located <u>outside</u> a closed surface. The number of lines entering the surface equals the number leaving the surface.



The net electric flux through any closed surface depends only on the charge <u>inside</u> that surface. The net flux through surface S is q_1/ϵ_0, the net flux through surface S' is $(q_2 + q_3)/\epsilon_0$, and the net flux through surface S'' is zero. Charge q_4 does not contribute to the flux through any surface because it is outside all surfaces.

Example 24.03

A spherical gaussian surface surrounds a point charge q. Describe what happens to the total flux through the surface if: (a) the charge is tripled, (b) the radius of the sphere is doubled, (c) the surface is changed to a cube, and (d) the charge is moved to another location inside the surface.

(a) The flux through the surface is tripled because flux is proportional to the amount of charge inside the surface.

(b) The flux does not change because all electric field lines from the charge pass through the sphere, regardless of its radius.

(c) The flux does not change when the shape of the gaussian surface changes because all electric field lines from the charge pass through the surface, regardless of its shape.

(d) The flux does not change when the charge is moved to another location inside that surface because Gauss's law refers to the total charge enclosed, regardless of where the charge is located inside the surface.

Gauss's Law

Problem 24.09

The following charges are located inside a submarine: 5.00 μC , −9.00 μC, 27.00 μC, and −84.00 μC. (a) Calculate the net electric flux through the hull of the submarine. (b) Is the number of electric field lines leaving the submarine greater than, equal to, or less than the number entering it?

(a)

$$\Phi_E = \frac{q_{in}}{\epsilon_0}$$

$$\Phi_E = \frac{(5.00 \times 10^{-6}) + (-9.00 \times 10^{-6}) + (27.00 \times 10^{-6}) + (-84.00 \times 10^{-6})}{8.85 \times 10^{-12}}$$

$$\Phi_E = -6.89 \times 10^{6}\ \mathrm{N \cdot m^2/C}$$

(b) Since the net electric flux is negative, more lines enter than leave the surface.

Gauss's Law

Problem 24.21

A charge of 170 μC is at the center of a cube of edge 80.0 m. (a) Find the flux through the whole surface of the cube. (b) Find the total flux through each face of the cube. (c) What If? Would your answers to parts (a) or (b) change if the charge were not at the center? Explain.

(a) $\Phi_E = \dfrac{q_{in}}{\epsilon_0}$

$$\Phi_E = \frac{170 \times 10^{-6}}{8.85 \times 10^{-12}}$$

$$\Phi_E = 1.92 \times 10^{7}\ \mathrm{N \cdot m^2/C}$$

(b) $(\Phi_E)_{one\ face} = \dfrac{1}{6}\Phi_E = \dfrac{1.92 \times 10^{7}}{6} = 3.20 \times 10^{6}\ \mathrm{N \cdot m^2/C}$

(c) The answer to (a) would remain the same, since the overall flux would remain the same. The answer to (b) would change because the flux through each face of the cube would not be equal with an asymmetric charge distribution. The sides of the cube nearer the charge would have more flux and the ones further away would have less.

Application of Gauss’s Law to Various Charge Distributions

- Gauss’s law is useful in determining electric fields when the charge distribution is characterized by a high degree of symmetry.

- The closed surface should satisfy one or more of the following conditions:

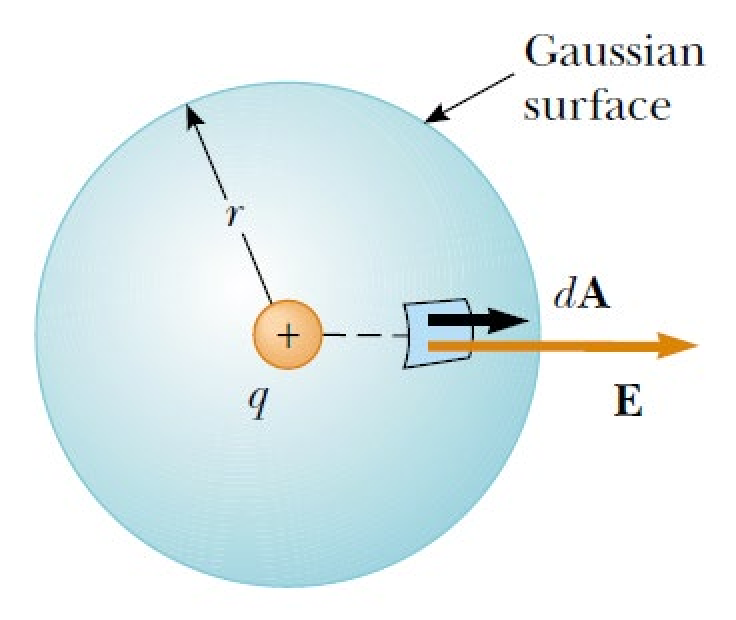
  1. The value of the electric field can be argued by symmetry to be constant over the surface.
  2. The dot product in Gauss’s law can be expressed as a simple algebraic product $E \cdot dA$ because $\vec{E}$ and $d\vec{A}$ are parallel.
  3. The dot product in Gauss’s law is zero because $\vec{E}$ and $d\vec{A}$ are perpendicular.
  4. The field can be argued to be zero over the surface.

Example 24.04

Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

A single charge represents the simplest possible charge distribution, and we use this familiar case to show how to solve for the electric field with Gauss's law. The following figure and our discussion of the electric field due to a point charge in Chapter 23 help us to conceptualize the physical situation. Because the space around the single charge has spherical symmetry, we categorize this problem as one in which there is enough symmetry to apply Gauss's law. To analyze any Gauss's law problem, we consider the details of the electric field and choose a gaussian surface that satisfies some or all of the conditions that we have listed above. We choose a spherical gaussian surface of radius r centered on the point charge, as shown in the following figure. The electric field due to a positive point charge is directed radially outward by symmetry and is therefore normal to the surface at every point. Thus, as in condition (2), $\vec{E}$ is parallel to $d\vec{A}$ at each point. Therefore, $\vec{E} \cdot d\vec{A} = E\, dA$ and Gauss's law gives:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E\, dA = \frac{q}{\epsilon_0}$$



The point charge q is at the center of the spherical gaussian surface, and $\vec{E}$ is parallel to $d\vec{A}$ at every point on the surface.

Example 24.04

Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

By symmetry, E is constant everywhere on the surface, which satisfies condition (1), so it can be removed from the integral. Therefore:

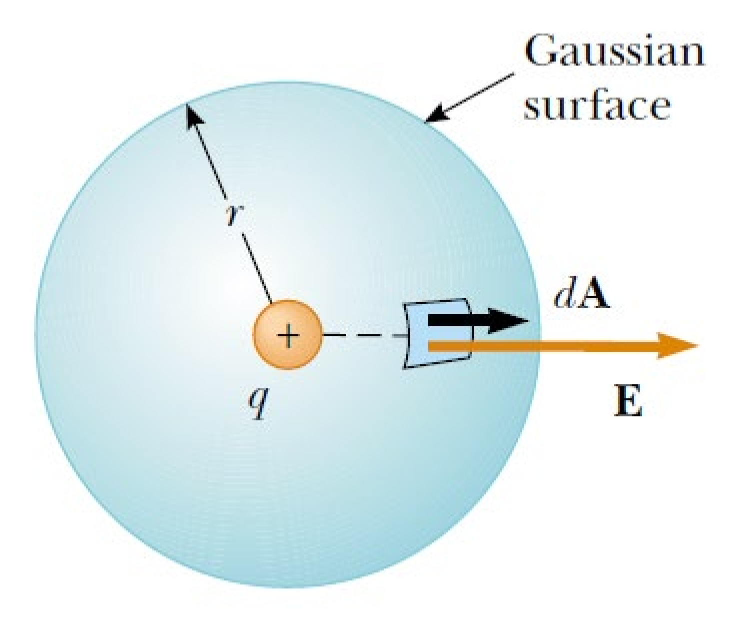
$$\oint E\, dA = E(4\pi r^2) = \frac{q}{\epsilon_0}$$

where we have used the fact that the surface area of a sphere is $4\pi r^2$. Now, we solve for the electric field:

$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

$$E = k_e \frac{q}{r^2}$$

To finalize this problem, note that this is the familiar electric field due to a point charge that we developed from Coulomb's law in Chapter 23.



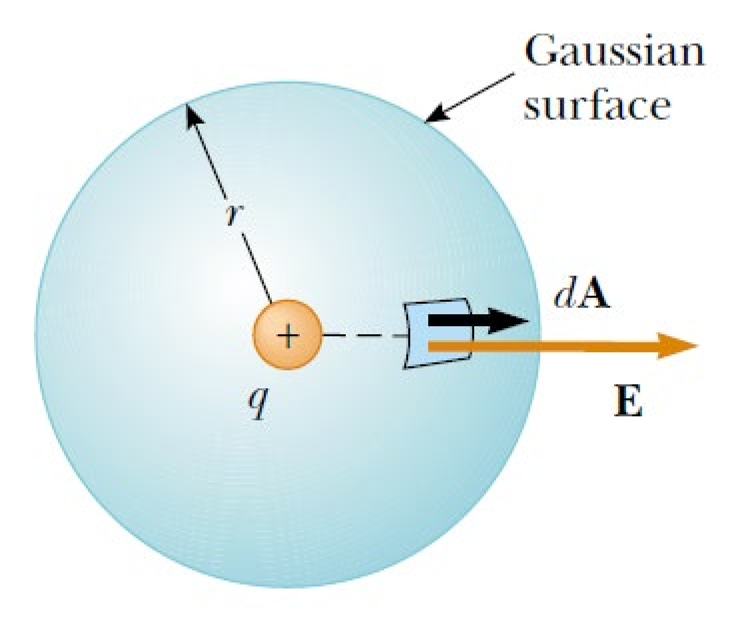
The point charge q is at the center of the spherical gaussian surface, and $\vec{E}$ is parallel to $d\vec{A}$ at every point on the surface.

Example 24.04

Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

What If? What if the charge in the following figure were not at the center of the spherical gaussian surface?

Answer In this case, while Gauss's law would still be valid, the situation would not possess enough symmetry to evaluate the electric field. Because the charge is not at the center, the magnitude of $\vec{E}$ would vary over the surface of the sphere and the vector $\vec{E}$ would not be everywhere perpendicular to the surface.



The point charge q is at the center of the spherical gaussian surface, and $\vec{E}$ is parallel to $d\vec{A}$ at every point on the surface.

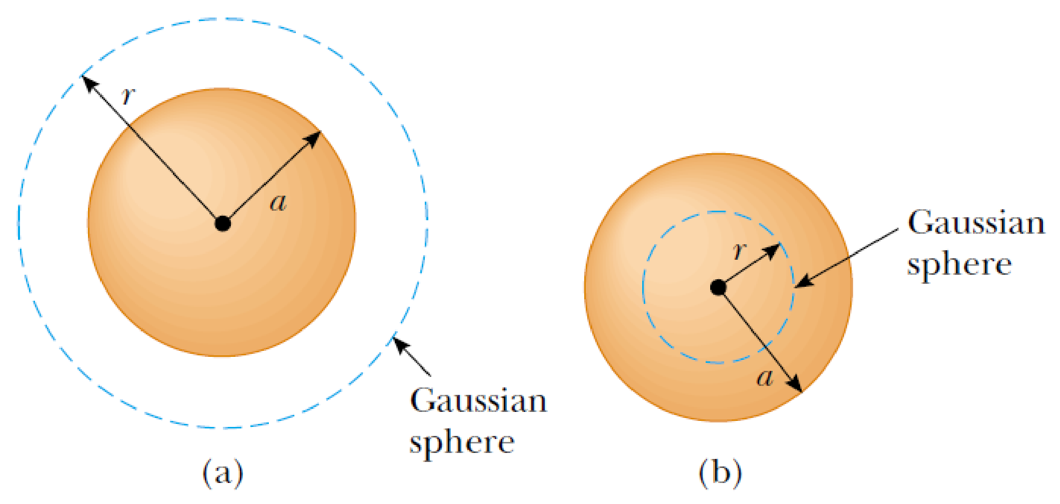
Example 24.05

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (see the figure below). Find the magnitude of the electric field at a point (a) outside the sphere, (b) inside the sphere.

(a) Because the charge distribution is spherically symmetric, we again select a spherical gaussian surface of radius r, concentric with the sphere, as shown in Fig. (a). For this choice, conditions (1) and (2) are satisfied, as they were for the point charge in Example 24.4. Following the line of reasoning given in Example 24.4, we find that:

$$E = k_e \frac{Q}{r^2} \qquad (\text{for } r > a)$$

Note that this result is identical to the one we obtained for a point charge. Therefore, we conclude that, <u>for a uniformly charged sphere, the field in the region external to the sphere is *equivalent* to that of a point charge located at the center of the sphere.</u>



A uniformly charged insulating sphere of radius a and total charge Q. (a) For points outside the sphere, a large spherical gaussian surface is drawn concentric with the sphere. In diagrams such as this, the dotted line represents the intersection of the gaussian surface with the plane of the page. (b) For points inside the sphere, a spherical gaussian surface smaller than the sphere is drawn.

جامعة الملك سعود King Saud University

Example 24.05

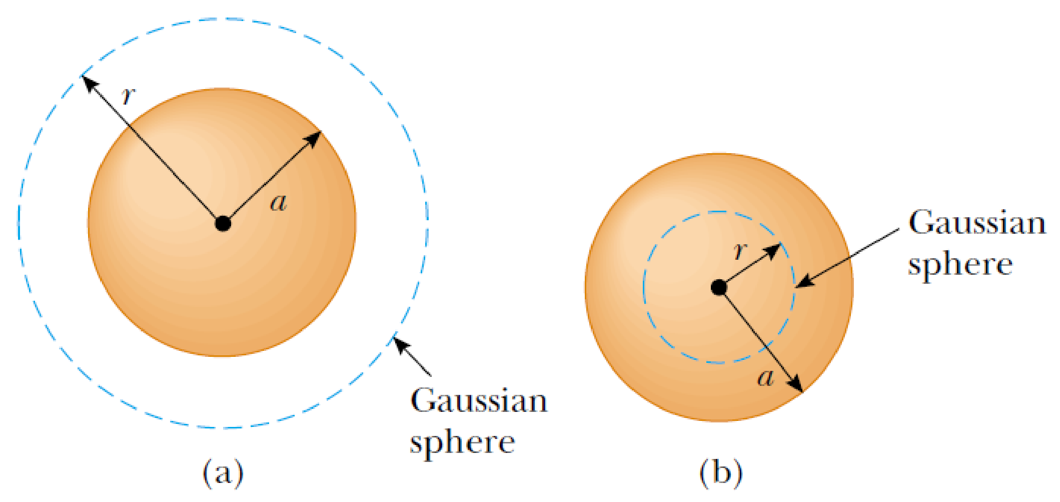
An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (see the figure below). Find the magnitude of the electric field at a point (a) outside the sphere, (b) inside the sphere.

(b) In this case we select a spherical gaussian surface having radius $r < a$, concentric with the insulating sphere (Fig. (b)). Let us denote the volume of this smaller sphere by V'. To apply Gauss's law in this situation, it is important to recognize that the charge q_{in} within the gaussian surface of volume V' is less than Q. To calculate q_{in}, we use the fact that $q_{in} = \rho V'$:

$$q_{in} = \rho V' = \rho\left(\frac{4}{3}\pi r^3\right)$$

By symmetry, the magnitude of the electric field is constant everywhere on the spherical gaussian surface and is normal to the surface at each point—both conditions (1) and (2) are satisfied. Therefore, Gauss's law in the region $r < a$ gives:

$$\oint E\,dA = E\oint dA = E(4\pi r^2) = \frac{q_{in}}{\epsilon_0}$$



A uniformly charged insulating sphere of radius a and total charge Q. (a) For points outside the sphere, a large spherical gaussian surface is drawn concentric with the sphere. In diagrams such as this, the dotted line represents the intersection of the gaussian surface with the plane of the page. (b) For points inside the sphere, a spherical gaussian surface smaller than the sphere is drawn.

Example 24.05

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (see the figure below). Find the magnitude of the electric field at a point (a) outside the sphere, (b) inside the sphere.

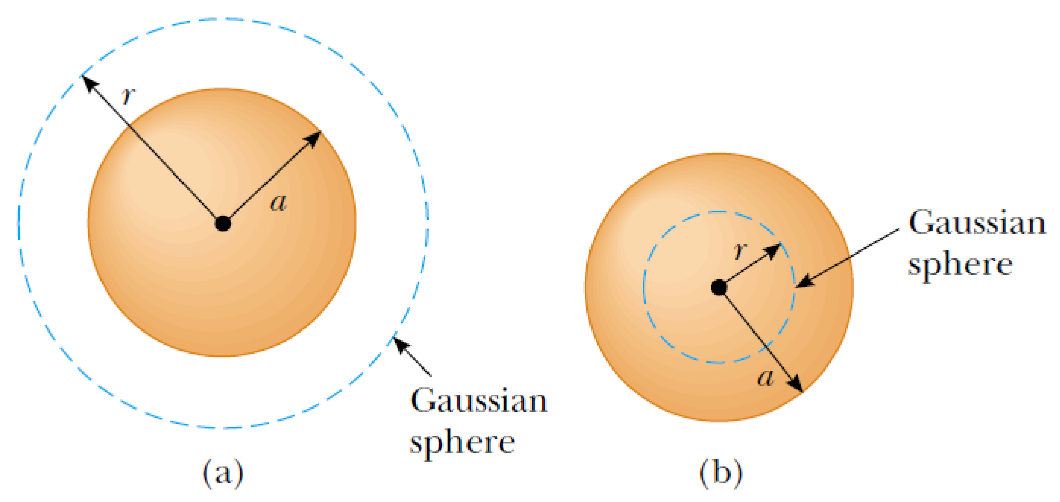
(b) Solving for E gives:

$$E = \frac{q_{in}}{4\pi\epsilon_0 r^2} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0} r$$

Because $\rho = Q/\left(\frac{4}{3}\pi a^3\right)$ by definition and because $k_e = 1/(4\pi\epsilon_0)$, this expression for E can be written as:

$$E = \frac{Qr}{4\pi\epsilon_0 a^3} = k_e \frac{Q}{a^3} r \qquad (\text{for } r < a)$$

Note that this result for E differs from the one we obtained in part (a). It shows that $E \to 0$ as $r \to 0$. Therefore, the result eliminates the problem that would exist at $r = 0$ if E varied as $1/r^2$ inside the sphere as it does outside the sphere. That is, if $E \propto 1/r^2$ for $r < a$, the field would be infinite at $r = 0$, which is physically impossible.



A uniformly charged insulating sphere of radius a and total charge Q. (a) For points outside the sphere, a large spherical gaussian surface is drawn concentric with the sphere. In diagrams such as this, the dotted line represents the intersection of the gaussian surface with the plane of the page. (b) For points inside the sphere, a spherical gaussian surface smaller than the sphere is drawn.

Example 24.05

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (see the figure below). Find the magnitude of the electric field at a point (a) outside the sphere, (b) inside the sphere.

Suppose we approach the radial position $r = a$ from inside the sphere and from outside. Do we measure the same value of the electric field from both directions?

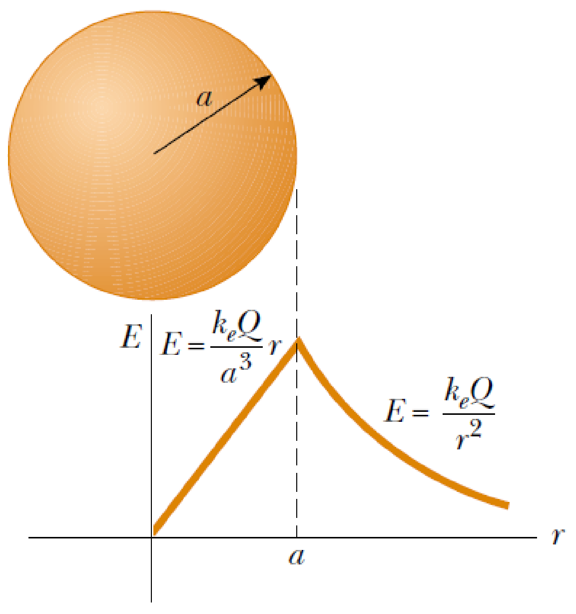
From (a), we see that the field approaches a value from the outside given by:

$$E = \lim_{r \to a}\left(k_e \frac{Q}{r^2}\right) = k_e \frac{Q}{a^2}$$

From the inside, (b) gives us:

$$E = \lim_{r \to a}\left(k_e \frac{Q}{a^3} r\right) = k_e \frac{Q}{a^3} a = k_e \frac{Q}{a^2}$$

Thus, the value of the field is the same as we approach the surface from both directions. A plot of E versus r is shown in the figure below. Note that the magnitude of the field is continuous, but the derivative of the field magnitude is not.



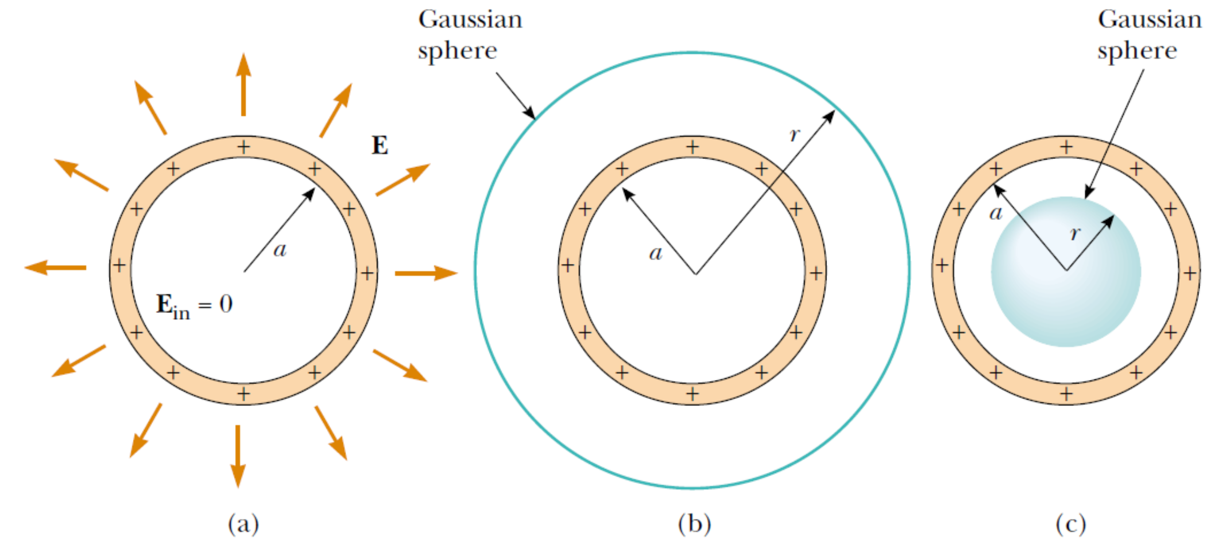
A plot of E versus r for a uniformly charged insulating sphere. The electric field inside the sphere ($r < a$) varies linearly with r. The field outside the sphere ($r > a$) is the same as that of a point charge Q located at $r = 0$.

Example 24.06

A thin spherical shell of radius a has a total charge Q distributed uniformly over its surface (see Fig. (a)). Find the electric field at points (a) outside and (b) inside the shell.

(a) The calculation for the field outside the shell is identical to that for the solid sphere shown in Example 24.5a. If we construct a spherical gaussian surface of radius $r > a$ concentric with the shell (see Fig. (b)), the charge inside this surface is Q. Therefore, the field at a point outside the shell is equivalent to that due to a point charge Q located at the center:

$$E = k_e \frac{Q}{r^2} \qquad (\text{for } \; r > a)$$

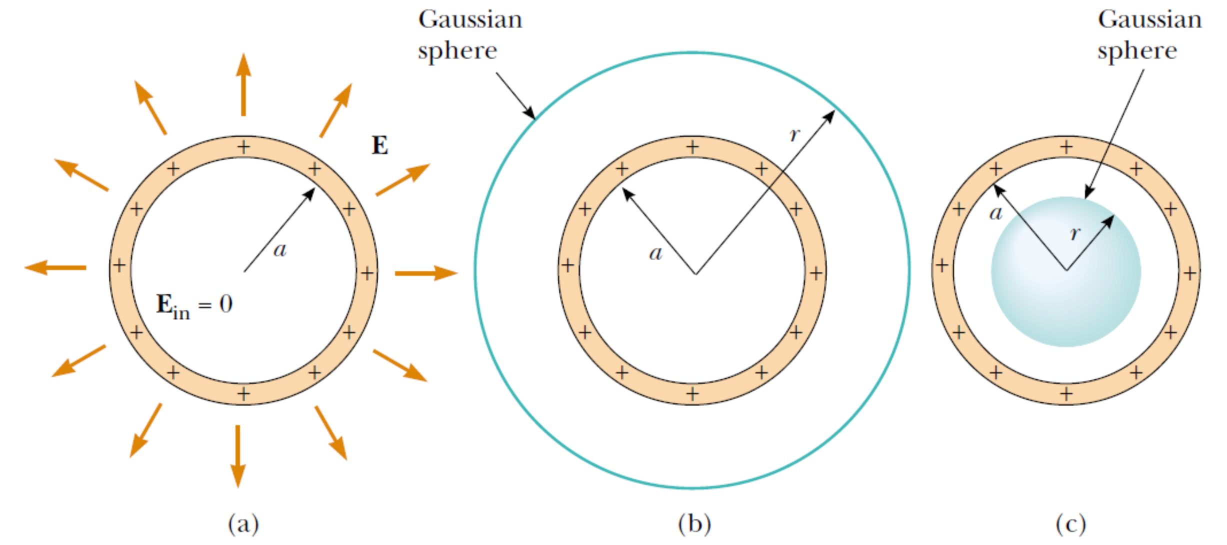


(a) The electric field inside a uniformly charged spherical shell is zero. The field outside is the same as that due to a point charge Q located at the center of the shell. (b) Gaussian surface for.$r > a$. (c) Gaussian surface for $r < a$.

Example 24.06

A thin spherical shell of radius a has a total charge Q distributed uniformly over its surface (see Fig. (a)). Find the electric field at points (a) outside and (b) inside the shell.

(b) The electric field inside the spherical shell is zero. This follows from Gauss's law applied to a spherical surface of radius $r < a$ concentric with the shell (see Fig. (c)). Because of the spherical symmetry of the charge distribution and because the net charge inside the surface is zero satisfaction of conditions (1) and (2) again—application of Gauss's law shows that $E = 0$ in the region $r < a$. We obtain the same results using $\vec{E} = k_e \int \frac{dq}{r^2} \hat{r}$ and integrating over the charge distribution. This calculation is rather complicated. Gauss's law allows us to determine these results in a much simpler way.

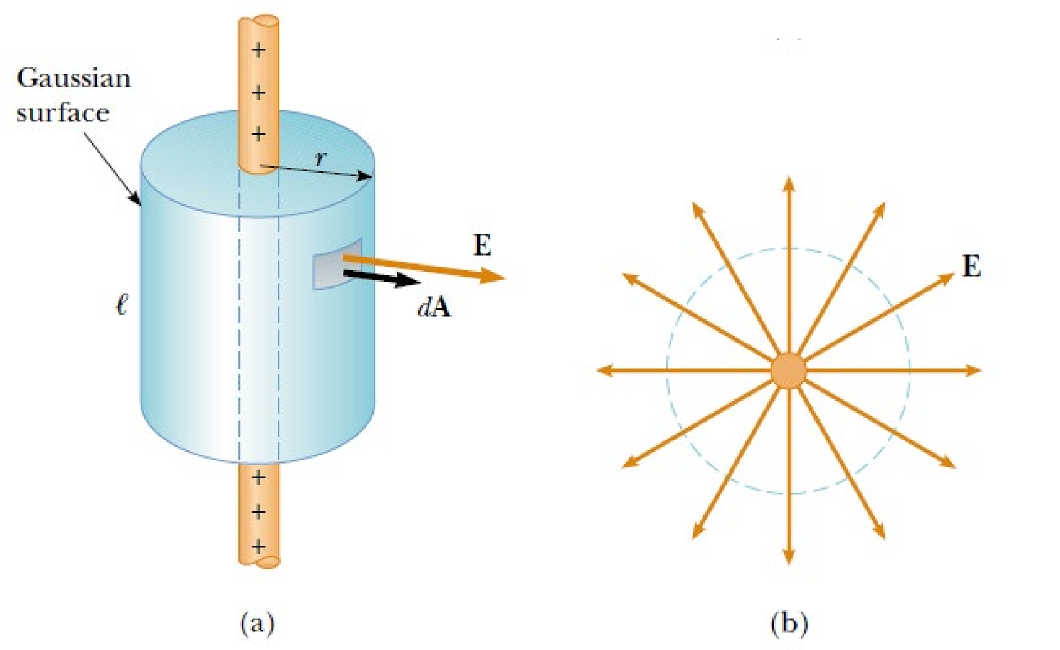


(a) The electric field inside a uniformly charged spherical shell is zero. The field outside is the same as that due to a point charge Q located at the center of the shell. (b) Gaussian surface for.$r > a$. (c) Gaussian surface for $r < a$.

Example 24.07

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ (see Fig. (a)).

The symmetry of the charge distribution requires that $\vec{E}$ be perpendicular to the line charge and directed outward, as shown in Fig. (a) and (b). To reflect the symmetry of the charge distribution, we select a cylindrical gaussian surface of radius r and length l that is coaxial with the line charge. For the curved part of this surface, $\vec{E}$ is constant in magnitude and perpendicular to the surface at each point—satisfaction of conditions (1) and (2). Furthermore, the flux through the ends of the gaussian cylinder is zero because $\vec{E}$ is parallel to these surfaces—the first application we have seen of condition (3).
We take the surface integral in Gauss's law over the entire gaussian surface. Because of the zero value of $\vec{E} \cdot d\vec{A}$ for the ends of the cylinder, however, we can restrict our attention to only the curved surface of the cylinder.



(a) An infinite line of charge surrounded by a cylindrical gaussian surface concentric with the line. (b) An end view shows that the electric field at the cylindrical surface is constant in magnitude and perpendicular to the surface.

King Saud University

Example 24.07

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ (see Fig. (a)).

The total charge inside our gaussian surface is λl. Applying Gauss's law and conditions (1) and (2), we find that for the curved surface:

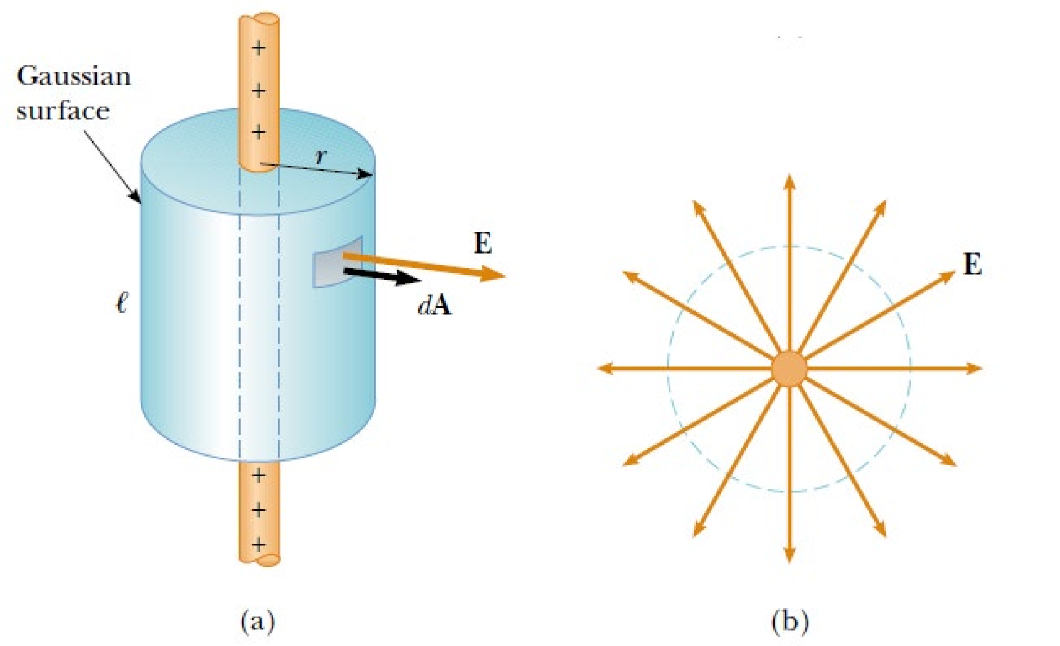
$$\oint E\,dA = E\oint dA = EA = \frac{q_{in}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

The area of the curved surface is $A = 2\pi r l$; therefore:

$$E(2\pi r l) = \frac{\lambda l}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{2k_e}{r}\lambda$$

Thus, we see that the electric field due to a cylindrically symmetric charge distribution varies as $1/r$, whereas the field external to a spherically symmetric charge distribution varies as $1/r^2$. $E = 2k_e\frac{\lambda}{r}$ was also derived by integration of the field of a point charge.



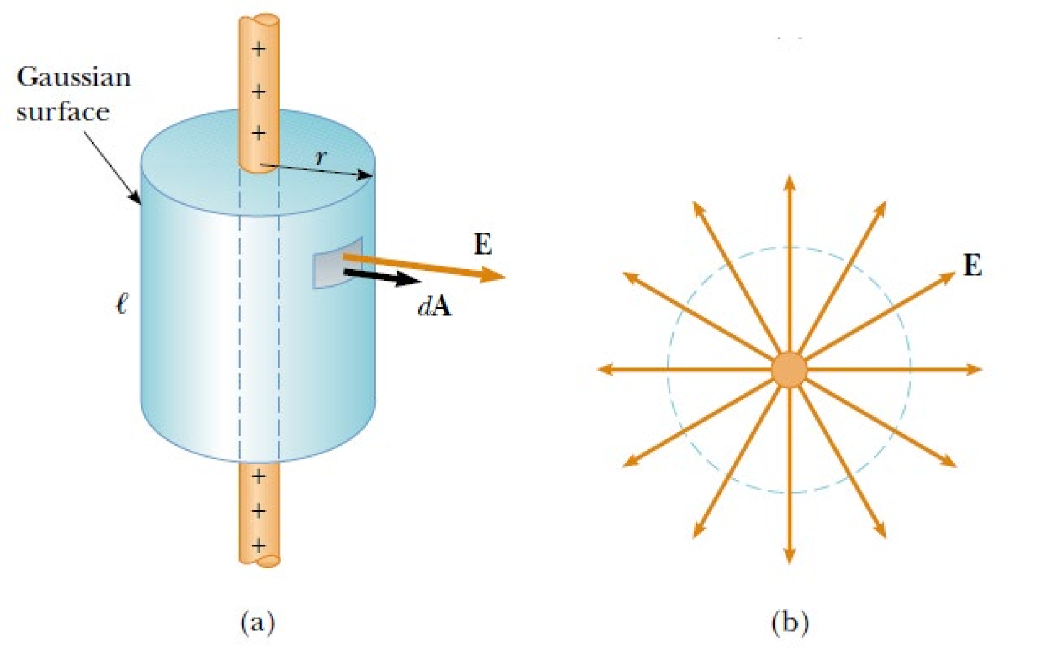
(a) An infinite line of charge surrounded by a cylindrical gaussian surface concentric with the line. (b) An end view shows that the electric field at the cylindrical surface is constant in magnitude and perpendicular to the surface.

Example 24.07

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ (see Fig. (a)).

What if the line segment in this example were not infinitely long?

If the line charge in this example were of finite length, the result for E would not be that given by $E = 2k_e \frac{\lambda}{r}$. A finite line charge does not possess sufficient symmetry for us to make use of Gauss's law. This is because the magnitude of the electric field is no longer constant over the surface of the gaussian cylinder—the field near the ends of the line would be different from that far from the ends. Thus, condition (1) would not be satisfied in this situation. Furthermore, $\vec{E}$ is not perpendicular to the cylindrical surface at all points—the field vectors near the ends would have a component parallel to the line. Thus, condition (2) would not be satisfied. For points close to a finite line charge and far from the ends, $E = 2k_e \frac{\lambda}{r}$ gives a good approximation of the value of the field.



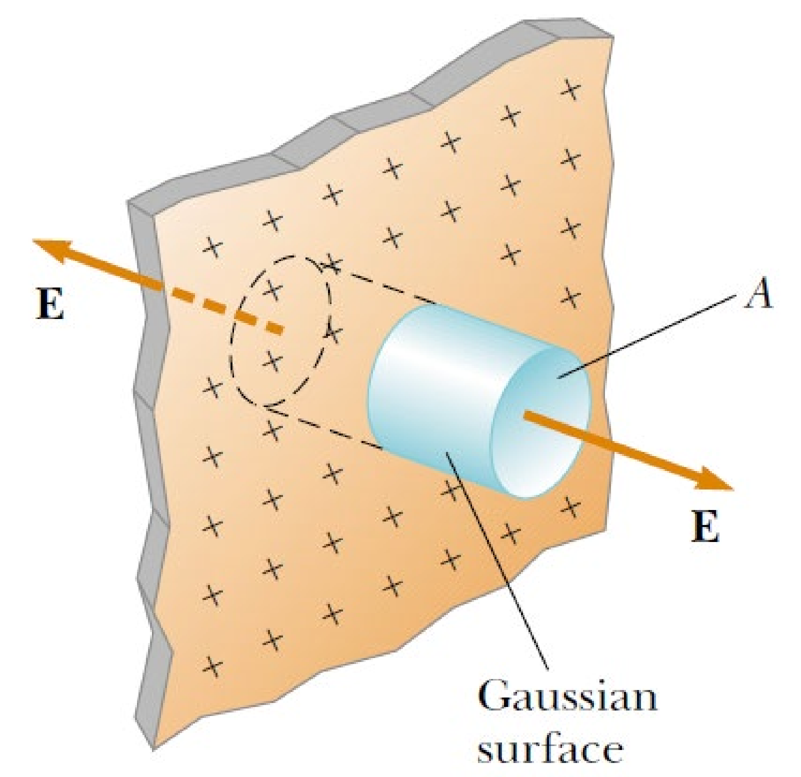
(a) An infinite line of charge surrounded by a cylindrical gaussian surface concentric with the line. (b) An end view shows that the electric field at the cylindrical surface is constant in magnitude and perpendicular to the surface.

جامعة الملك سعود
King Saud University
1957

Example 24.08

Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.

By symmetry, $\vec{E}$ must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane. The fact that the direction of $\vec{E}$ is away from positive charges indicates that the direction of $\vec{E}$ on one side of the plane must be opposite its direction on the other side, as shown in the following figure. A gaussian surface that reflects the symmetry is a small cylinder whose axis is perpendicular to the plane and whose ends each have an area A and are equidistant from the plane. Because $\vec{E}$ is parallel to the curved surface—and, therefore, perpendicular to $d\vec{A}$ everywhere on the surface—condition (3) is satisfied and there is no contribution to the surface integral from this surface. For the flat ends of the cylinder, conditions (1) and (2) are satisfied. The flux through each end of the cylinder is EA; hence, the total flux through the entire gaussian surface is just that through the ends, $\Phi_E = 2EA$.



A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is EA through each end of the gaussian surface and zero through its curved surface.

King Saud University

Example 24.08

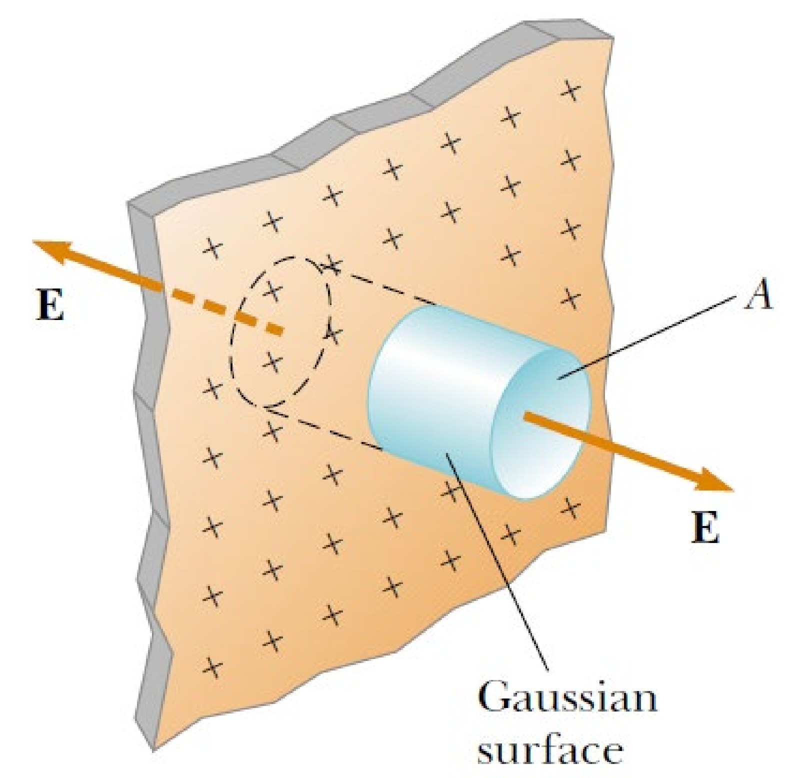
Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.

Noting that the total charge inside the surface is $q_{in} = \sigma A$, we use Gauss's law and find that the total flux through the gaussian surface is:

$$\Phi_E = 2EA = \frac{q_{in}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

Because the distance from each flat end of the cylinder to the plane does not appear in $E = \sigma/2\epsilon_0$, we conclude that $E = \sigma/2\epsilon_0$ at any distance from the plane. That is, the field is uniform everywhere.



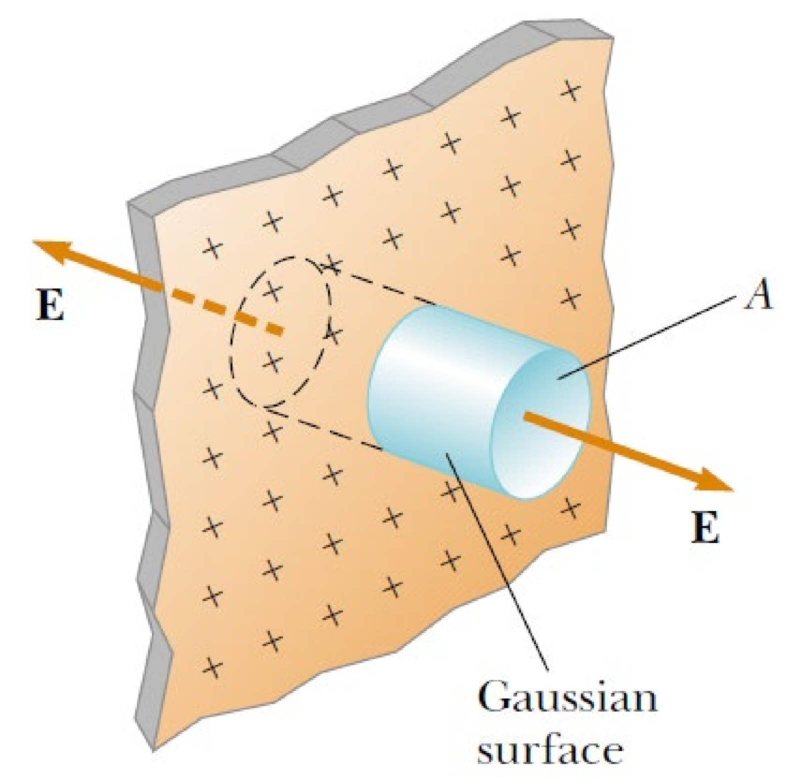
A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is EA through each end of the gaussian surface and zero through its curved surface.

Example 24.08

Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.

Suppose we place two infinite planes of charge parallel to each other, one positively charged and the other negatively charged. Both planes have the same surface charge density. What does the electric field look like now?

In this situation, the electric fields due to the two planes add in the region between the planes, resulting in a uniform field of magnitude σ/ϵ_0, and cancel elsewhere to give a field of zero. This is a practical way to achieve uniform electric fields, such as those needed in the CRT tube.



A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is EA through each end of the gaussian surface and zero through its curved surface.

Problem 24.24

A solid sphere of radius 40.0 cm has a total positive charge of 26.0 μC uniformly distributed throughout its volume. Calculate the magnitude of the electric field (a) 0 cm, (b) 10.0 cm, (c) 40.0 cm, and (d) 60.0 cm from the center of the sphere.

(a) $E = \dfrac{k_e Q r}{a^3} = 0$

(b) $E = \dfrac{k_e Q r}{a^3} = \dfrac{8.99 \times 10^9 \times 26.0 \times 10^{-6} \times 0.1}{(0.4)^3} = 365\ \mathrm{N/C}$

(c) $E = \dfrac{k_e Q}{r^2} = \dfrac{8.99 \times 10^9 \times 26.0 \times 10^{-6}}{(0.4)^2} = 1.46 \times 10^6\ \mathrm{N/C}$

(d) $E = \dfrac{k_e Q}{r^2} = \dfrac{8.99 \times 10^9 \times 26.0 \times 10^{-6}}{(0.6)^2} = 6.49 \times 10^5\ \mathrm{N/C}$

The direction for each electric field is radially outward.

Problem 24.31

Consider a thin spherical shell of radius 14.0 cm with a total charge of 32.0 μC distributed uniformly on its surface. Find the electric field (a) 10.0 cm and (b) 20.0 cm from the center of the charge distribution.

(a) $E = 0$

(b) $E = \frac{k_e Q}{r^2} = \frac{8.99 \times 10^9 \times 32.0 \times 10^{-6}}{(0.2)^2} = 7.19 \times 10^6 \text{ N/C}$

The direction of the electric field is radially outward.

Problem 24.35

A uniformly charged, straight filament 7.00 m in length has a total positive charge of 2.00 μC . An uncharged cardboard cylinder 2.00 cm in length and 10.0 cm in radius surrounds the filament at its center, with the filament as the axis of the cylinder. Using reasonable approximations, find (a) the electric field at the surface of the cylinder and (b) the total electric flux through the cylinder.

(a)

$$E = \frac{2k_e}{r}\lambda$$

$$E = \frac{2 \times 8.99 \times 10^9}{0.1} \times \frac{2.00 \times 10^{-6}}{7.00}$$

$$E = 5.14 \times 10^4 \text{ N/C}$$, radially outward

(b) $\Phi_E = EA\cos\theta$

$$\Phi_E = E(2\pi r l)\cos(0)$$

$$\Phi_E = 5.14 \times 10^4 \times 2 \times 3.14 \times 0.1 \times 0.02 \times 1$$

$$\Phi_E = 646 \text{ N} \cdot \text{m}^2/\text{C}$$

King Saud University
جامعة الملك سعود

Problem 24.37

A large flat horizontal sheet of charge has a charge per unit area of 9.00 $\mu C/m^2$. Find the electric field just above the middle of the sheet.

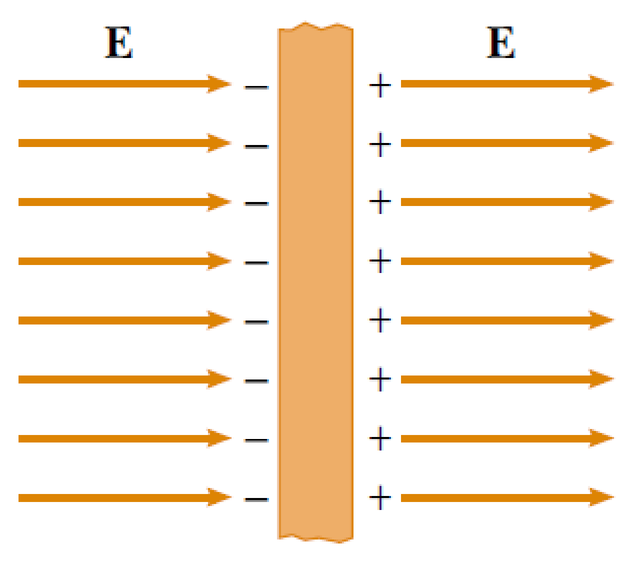
$$E = \frac{\sigma}{2\epsilon_0}$$

$$E = \frac{9.00 \times 10^{-6}}{2(8.85 \times 10^{-12})}$$

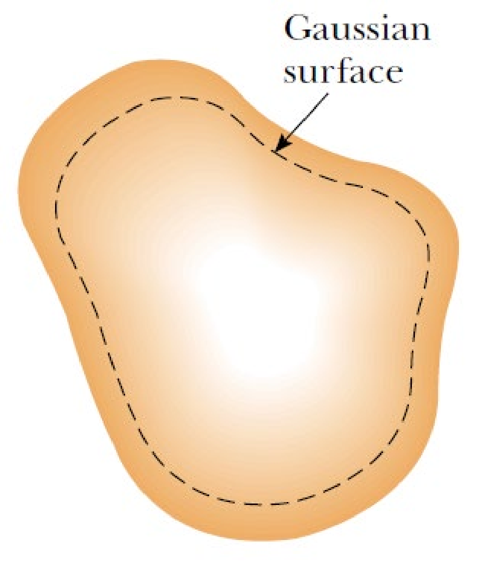
$E = 5.08 \times 10^5$ N/C , upward

Conductors in Electrostatic Equilibrium

- When there is no net motion of charge within a conductor, the conductor is in electrostatic equilibrium.
- A conductor in electrostatic equilibrium has the following properties:
  1. The electric field is zero everywhere inside the conductor.
  2. If an isolated conductor carries a charge, the charge resides on its surface.



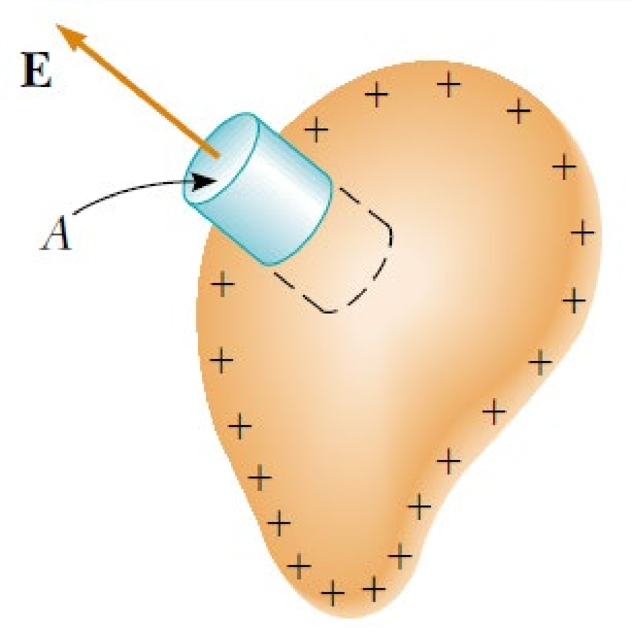
A conducting slab in an external electric field $\vec{E}$. The charges induced on the two surfaces of the slab produce an electric field that opposes the external field, giving a resultant field of zero inside the slab.



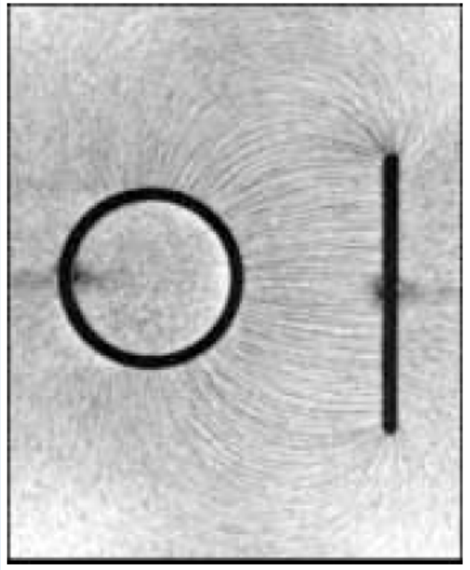
A conductor of arbitrary shape. The broken line represents a gaussian surface that can be as close to the surface of the conductor as we wish.

Conductors in Electrostatic Equilibrium

- When there is no net motion of charge within a conductor, the conductor is in electrostatic equilibrium.
- A conductor in electrostatic equilibrium has the following properties:
  1. The electric field is zero everywhere inside the conductor.
  2. If an isolated conductor carries a charge, the charge resides on its surface.
  3. The electric field just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude σ/ϵ_0, where σ is the surface charge density at that point.
  4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.



A gaussian surface in the shape of a small cylinder is used to calculate the electric field just outside a charged conductor. The flux through the gaussian surface is EA. Remember that $\vec{E}$ is zero inside the conductor.



Electric field pattern surrounding a charged conducting plate placed near an oppositely charged conducting cylinder. Small pieces of thread suspended in oil align with the electric field lines. Note that (1) the field lines are perpendicular to both conductors and (2) there are no lines inside the cylinder ($E = 0$).

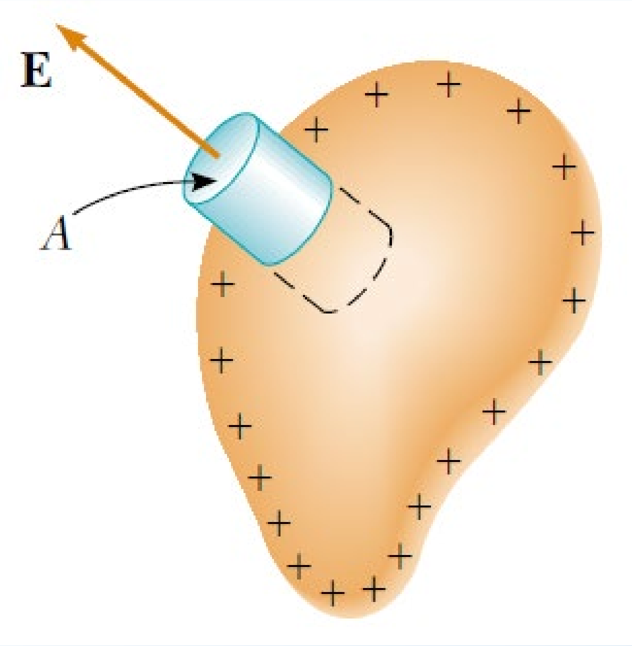
Conductors in Electrostatic Equilibrium

- Applying Gauss's law to surface A, we obtain:

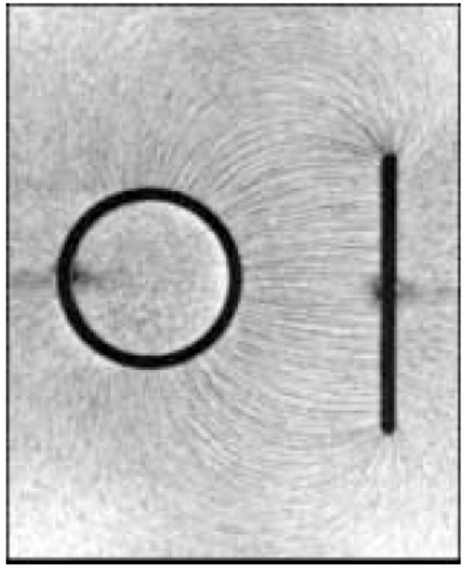
$$\oint E\,dA = E\oint dA = EA = \frac{q_{in}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

- Where we have used the fact that $q_{in} = \sigma A$. Solving for E gives for the electric field just outside a charged conductor:

$$E = \frac{\sigma}{\epsilon_0}$$



A gaussian surface in the shape of a small cylinder is used to calculate the electric field just outside a charged conductor. The flux through the gaussian surface is EA. Remember that $\vec{E}$ is zero inside the conductor.



Electric field pattern surrounding a charged conducting plate placed near an oppositely charged conducting cylinder. Small pieces of thread suspended in oil align with the electric field lines. Note that (1) the field lines are perpendicular to both conductors and (2) there are no lines inside the cylinder ($E = 0$).

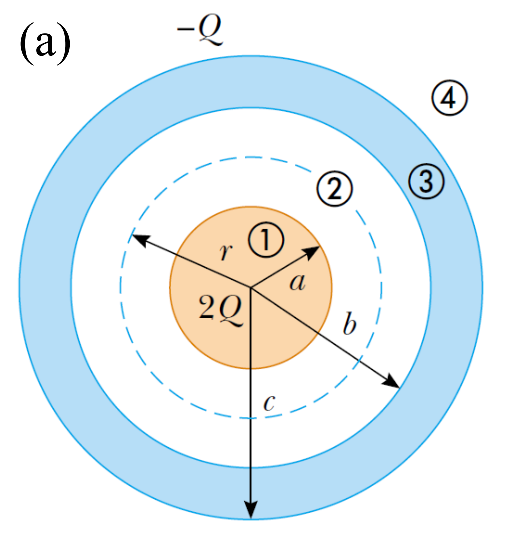
Example 24.10

A solid conducting sphere of radius a carries a net positive charge $2Q$. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge $-Q$. Using Gauss's law, find the electric field in the regions labeled ①, ②, ③, and ④ in the following figure and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

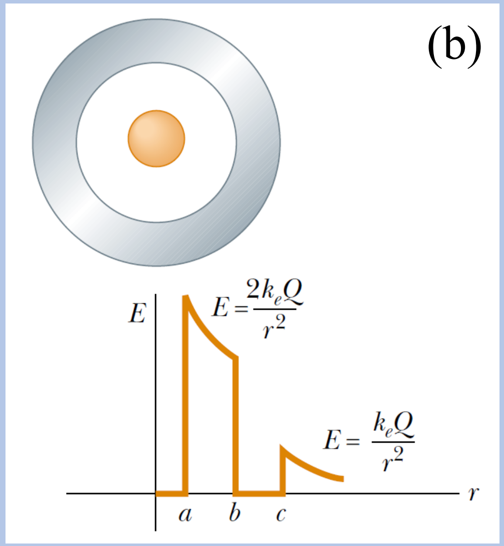
First note that the charge distributions on both the sphere and the shell are characterized by spherical symmetry around their common center. To determine the electric field at various distances r from this center, we construct a spherical gaussian surface for each of the four regions of interest. Such a surface for region ② is shown in Fig. (a).

To find E inside the solid sphere (region ①), consider a gaussian surface of radius $r < a$. Because there can be no charge inside a conductor in electrostatic equilibrium, we see that $q_{in} = 0$; thus, on the basis of Gauss's law and symmetry, $E_1 = 0$ for $r < a$.

In region ②—between the surface of the solid sphere and the inner surface of the shell—we construct a spherical gaussian surface of radius r where $a < r < b$ and note that the charge inside this surface is $+2Q$ (the charge on the solid sphere). Because of the spherical symmetry, the electric field lines must be directed radially outward and be constant in magnitude on the gaussian surface.



A solid conducting sphere of radius a and carrying a charge $2Q$ surrounded by a conducting spherical shell carrying a charge $-Q$.



A plot of E versus r for the two-conductor system

Example 24.10

A solid conducting sphere of radius a carries a net positive charge $2Q$. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge $-Q$. Using Gauss's law, find the electric field in the regions labeled ①, ②, ③, and ④ in the following figure and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

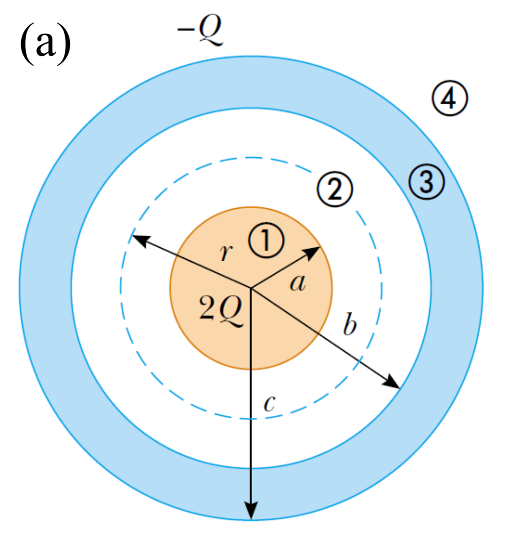
Following Example 24.4 and using Gauss's law, we find that:

$$E_2 A = E_2(4\pi r^2) = \frac{q_{in}}{\epsilon_0} = \frac{2Q}{\epsilon_0}$$

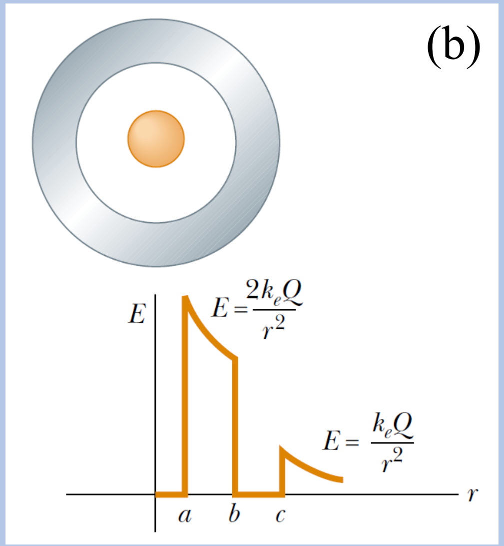
$$E_2 = \frac{2Q}{4\pi\epsilon_0 r^2} = \frac{2k_e Q}{r^2} \qquad (for\ a < r < b)$$

In region ④, where $r > c$, the spherical gaussian surface we construct surrounds a total charge of $q_{in} = 2Q + (-Q) = Q$. Therefore, application of Gauss's law to this surface gives:

$$E_4 = \frac{k_e Q}{r^2} \qquad (for\ r > c)$$



A solid conducting sphere of radius a and carrying a charge $2Q$ surrounded by a conducting spherical shell carrying a charge $-Q$.



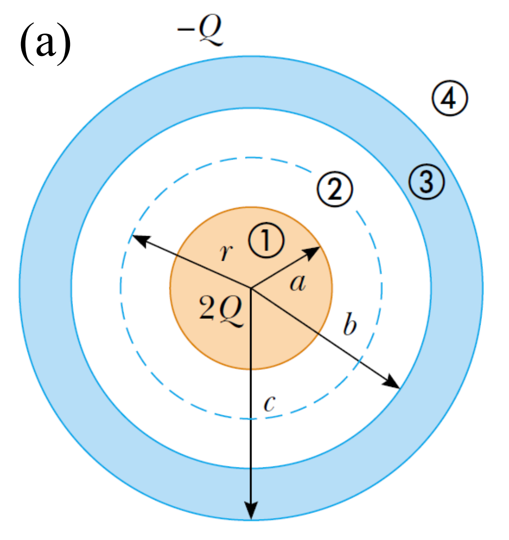
A plot of E versus r for the two-conductor system

King Saud University

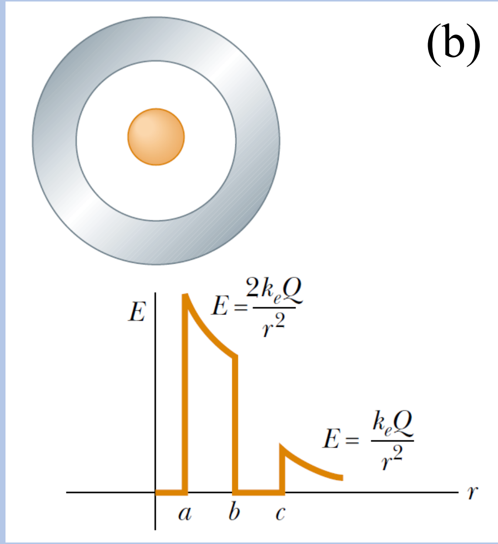
Example 24.10

A solid conducting sphere of radius a carries a net positive charge $2Q$. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge $-Q$. Using Gauss's law, find the electric field in the regions labeled ①, ②, ③, and ④ in the following figure and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

In region ③, the electric field must be zero because the spherical shell is also a conductor in equilibrium. Fig. (b) shows a graphical representation of the variation of electric field with r.
If we construct a gaussian surface of radius r where $b < r < c$, we see that q_{in} must be zero because $E_3 = 0$. From this argument, we conclude that the charge on the inner surface of the spherical shell must be $-2Q$ to cancel the charge $+2Q$ on the solid sphere. Because the net charge on the shell is $-Q$, we conclude that its outer surface must carry a charge $+Q$.



A solid conducting sphere of radius a and carrying a charge $2Q$ surrounded by a conducting spherical shell carrying a charge $-Q$.



A plot of E versus r for the two-conductor system

جامعة الملك سعود
King Saud University

Problem 24.40

On a clear, sunny day, a vertical electric field of about 130 N/C points down over flat ground. What is the surface charge density on the ground for these conditions?

From Gauss's Law:

$$EA = \frac{Q}{\epsilon_0}$$

$$\sigma = \frac{Q}{A} = \epsilon_0 E$$

$$\sigma = \epsilon_0 E$$

$$\sigma = 8.85 \times 10^{-12} \times (-130)$$

$$\sigma = -1.15 \times 10^{-9}\ \mathrm{C/m^2}$$

$$\sigma = -1.15\ \mathrm{nC/m^2}$$

King Saud University

Problem 24.42

Additional problem

A solid copper sphere of radius 15.0 cm carries a charge of 40 nC. Find the electric field (a) 12.0 cm , (b) 17.0 cm , and (c) 75.0 cm from the center of the sphere. (d) What If? How would your answers change if the sphere were hollow?

(a) All of the charge sits on the surface of the copper sphere at radius 15.0 cm. The field inside is zero.

(b) The charged sphere creates field at exterior points as if it were a point charge at the center:

$$E = \frac{k_e q}{r^2} = \frac{8.99 \times 10^9 \times 40.0 \times 10^{-9}}{(0.17)^2} = 1.24 \times 10^4 \text{ N/C}$$

(c) $$E = \frac{k_e q}{r^2} = \frac{8.99 \times 10^9 \times 40.0 \times 10^{-9}}{(0.75)^2} = 639 \text{N/C}$$

(d) All three answers would be the same.

Summary

King Saud University جامعة الملك سعود 1957

Typical Electric Field Calculations Using Gauss's Law

Charge Distribution	Electric Field	Location
Insulating sphere of radius R, uniform charge density, and total charge Q	$k_e \frac{Q}{r^2}$	$r > R$
	$k_e \frac{Q}{R^2} r$	$r < R$
Thin spherical shell of radius R and total charge Q	$k_e \frac{Q}{r^2}$	$r > R$
	0	$r < R$
Line charge of infinite length and charge per unit length λ	$2k_e \frac{\lambda}{r}$	Outside the line
Infinite charged plane having surface charge density σ	$\frac{\sigma}{2\epsilon_0}$	Everywhere outside the plane
Conductor having surface charge density σ	$\frac{\sigma}{\epsilon_0}$	Just outside the conductor
	0	Inside the conductor

$$k_e \frac{Q}{R^3} r$$