College of Sciences
Department of Physics & Astronomy

كلية العلوم
قسم الفيزياء والفلك

Final Exam
Academic Year 1441 Hijri- Second Semester

Exam Information معلومات الامتحان			
Course name	Classical Electrodynamics		اسم المقرر
Course Code	PHYS 507		رمز المقرر
Exam Date	Tuesday 28th April 2020	05-09-1441	تاريخ الامتحان
Exam Time	13:00 – 19:00		وقت الامتحان
Exam Duration	360 minutes	360 دقيقة	مدة الامتحان
Classroom No.	Take Home Exam		رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

تعليمات عامة:

1. الالتزام بوقت بداية الاختبار المدرجة في البوابة الاكاديمية ونهاية الاختبار التي ستحدد من قبل عضو هيئة التدريس.
2. تسليم اجابة الاختبار دون تأخير .ومن يتأخر عن موعد التسليم لن يقبل منه.
3. تسليم الإجابة عبر البلاك بورد أو البريد الإلكتروني أو كليهما(إما مستند pdf أو صورة ممسوحة ضوئيًا.)
4. يجب أن يتم إتمام الاختبار بشكل فردي .ويُحظر عرضه أو مناقشته مع أي شخص آخر، بما في ذلك (على سبيل المثال لا الحصر) الطلاب الآخرين في نفس المقرر.
5. يمكن للطالب استخدام أي مادة متاحة يريدها، بما في ذلك العروض التقديمية ومذكرة المحاضرات والكتب والإنترنت، ولا يجب نسخ المعلومة كما هي ولكن تكتب حسب فهم الطالب وإلا ستعتبر إقتباسا يؤثر على درجة الطالب.
6. سوف يتم النظر في جميع الحالات الطلابية التي لم يتمكنوا من أداء الاختبار المنزلي بعقد اختباربديل لها في بداية الفصل الدراسي الأول.

هذا الجزء خاص بأستاذ المادة
This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1	CLO 1.1:			20
2	CLO 1.2:			
3	CLO 1.3:			

Section A: Please answer all the following questions

SECTION A

1. A spherical charge distribution has a density given by:

$$\rho = \begin{cases} \rho_0\left(1-\frac{r^2}{a^2}\right), & (r \le a) \\ 0 & (r > a) \end{cases}$$

(a) Calculate the total charge *Q*. (1 mark)

(b) Find the electric field inside the sphere. (2 marks)

(c) Find the electric field outside the sphere. (2 marks)

(d) Find the potential outside the sphere. Set infinity as reference point. (2 marks)

(e) Find the potential inside the sphere. Set infinity as reference point. (2 marks)

(f) At which radial distance *r* (inside the spherical region) the electric field has its maximum value? (1 marks)

(g) Find the electric energy of this distribution. (2 marks)

Solution

(a) $Q = \int \rho dV = 4\pi \int_{r=0}^{\infty} \rho r^2 dr = 4\pi \int_{r=0}^{a} \rho_0 \left(1-\frac{r^2}{a^2}\right) r^2 dr =$

$$4\pi\rho_0 \int_{r=0}^{a} \left(1-\frac{r^2}{a^2}\right) r^2 dr = 4\pi\rho_0 \frac{2}{15} a^3 = \frac{8}{15}\pi\rho_0 a^3$$

(b) The electric field is calculated with the help of Gauss' law. Assume first a spherical Gaussian surface of radius *r* such that *r* < *a.*

Then:

$$\int \mathbf{E}\cdot d\mathbf{S} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int_0^r \rho dV \Rightarrow$$

$$E4\pi r^2 = 4\pi \frac{1}{\varepsilon_0}\int_0^r \rho_0\left(1-\frac{r'^2}{a^2}\right) r'^2 dr' \Rightarrow$$

$$E = \frac{\rho_0}{\varepsilon_0 r^2}\left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]$$

And since the electric field is a vector it has a radial direction due to radial symmetry of the charge distribution. So

$$\mathbf{E} = \frac{\rho_0}{\varepsilon_0 r^2}\left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]\hat{\mathbf{r}} = \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3} - \frac{r^3}{5a^2}\right]\hat{\mathbf{r}}$$

(c) Again the electric field is calculated with the help of Gauss' law. Assume first a spherical Gaussian surface of radius *r* such that *r* >*a.*

Then:

$$\int \mathbf{E}\cdot d\mathbf{S} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow \int \mathbf{E}\cdot d\mathbf{S} = \frac{8}{15\varepsilon_0}\pi\rho_0 a^3 \Rightarrow$$

$$E4\pi r^2 = \frac{8}{15\varepsilon_0}\pi\rho_0 a^3 \Rightarrow$$

$$E = \frac{2\rho_0 a^3}{15\varepsilon_0}\frac{1}{r^2}$$

And since the electric field is a vector it has a radial direction due to radial symmetry of the charge distribution. So

$$\mathbf{E} = \frac{2\rho_0 a^3}{15\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

(d) For the electric potential at a point outside the sphere we have:

$$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^r \mathbf{E}\cdot (dr'\hat{\mathbf{r}})$$

$$V(r) = -\int_\infty^r \frac{1}{r^2}\hat{\mathbf{r}}\cdot (dr'\hat{\mathbf{r}})$$

$$V(r) = -\frac{2\rho_0 a^3}{15\varepsilon_0}\int_\infty^r \frac{1}{r'^2}dr'$$

$$V(r) = \frac{2\rho_0 a^3}{15\varepsilon_0}\frac{1}{r'}\Big|_\infty^r = \frac{2\rho_0 a^3}{15\varepsilon_0 r}$$

(e) For the electric potential at a position *r* inside the sphere we have:

$$V(r) = -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{r} \mathbf{E}\cdot (dr'\hat{\mathbf{r}})$$

We split the integral in two parts:

$$V(r) = -\int_{\infty}^{r} \mathbf{E}\cdot (dr'\hat{\mathbf{r}}) \Longrightarrow$$

$$V(r) = -\int_{\infty}^{a} \mathbf{E}_{out}\cdot (dr'\hat{\mathbf{r}}) - \int_{a}^{r} \mathbf{E}_{in}\cdot (dr'\hat{\mathbf{r}}) \Longrightarrow$$

$$V(r) = -\int_{\infty}^{a} \frac{2\rho_0 a^3}{15\varepsilon_0}\frac{1}{r'^2}dr' - \int_{a}^{r} \frac{\rho_0}{\varepsilon_0 r^2}\left[\frac{r'^3}{3} - \frac{r'^5}{5a^2}\right]dr' \Longrightarrow$$

$$V(r) = -\frac{2\rho_0 a^3}{15\varepsilon_0}\int_{\infty}^{a} \frac{1}{r'^2}dr' - \frac{\rho_0}{\varepsilon_0 r^2}\int_{a}^{r} \frac{1}{r^2}\left[\frac{r'^3}{3} - \frac{r'^5}{5a^2}\right]dr' \Longrightarrow$$

$$V(r) = \frac{2\rho_0 a^2}{15\varepsilon_0} - \frac{\rho_0}{\varepsilon_0}\left[\frac{1}{6}(r^2 - a^2) - \frac{1}{20}\left(\frac{r^4}{a^2} - a^2\right)\right]$$

(f) Inside the sphere the magnitude of the electric field is:

$$E = \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3} - \frac{r^3}{5a^2}\right]$$

The derivative of this field with respect to *r* is:

$$\frac{dE}{dr} = \frac{\rho_0}{\varepsilon_0}\left[\frac{1}{3} - \frac{3r^2}{5a^2}\right]$$

$$\frac{dE}{dr} = 0 \Rightarrow \frac{1}{3} - \frac{3r^2}{5a^2} = 0 \Rightarrow \frac{3r^2}{5a^2} = \frac{1}{3}$$

$$r^2 = \frac{5}{9}a^2 \Longrightarrow r = \frac{\sqrt{5}}{3}a$$

The second derivative is

$$\frac{d^2E}{dr^2} = -\frac{\rho_0}{\varepsilon_0}\frac{6r}{5a^2} < 0$$

This means that at the distance $r = \frac{\sqrt{5}}{3}a$ the field maximizes.

(g) The energy of the configuration is given by:

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 dV \Longrightarrow$$

$$W = \frac{\varepsilon_0}{2}\left[\int_{\text{inside}} E^2 dV + \int_{\text{outside}} E^2 dV\right] \Longrightarrow$$

$$W = \frac{\varepsilon_0}{2}\left[4\pi\int_0^a E_{\text{inside}}^2 r^2 dr + 4\pi\int_a^\infty E_{\text{outside}}^2 r^2 dr\right] \Longrightarrow$$

$$W = \frac{\varepsilon_0}{2}\left[4\pi\int_0^a \left[\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^3}{5a^2}\right)\right]^2 r^2 dr + 4\pi\int_a^\infty \left[\frac{2\rho_0 a^3}{15\varepsilon_0}\frac{1}{r^2}\right]^2 r^2 dr\right] \Longrightarrow$$

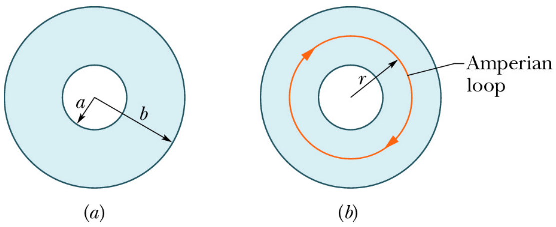
$$W = \frac{\varepsilon_0}{2}\left[4\pi\left(\frac{\rho_0}{\varepsilon_0}\right)^2\int_0^a \left[\left(\frac{r}{3} - \frac{r^3}{5a^2}\right)\right]^2 r^2 dr + 4\pi\left(\frac{2\rho_0 a^3}{15\varepsilon_0}\right)^2\int_a^\infty \frac{1}{r^2} dr\right] \Longrightarrow$$

$$W = \frac{\varepsilon_0}{2}\left[4\pi\left(\frac{\rho_0}{\varepsilon_0}\right)^2 \frac{4}{525}a^5 + 4\pi\left(\frac{2\rho_0 a^3}{15\varepsilon_0}\right)^2 \frac{1}{a}\right] \Longrightarrow$$

$$W = \frac{4\pi\rho_0^2 a^5}{2\varepsilon_0}\left[\frac{4}{525} + \frac{4}{225}\right] \Longrightarrow$$

$$W = \frac{16\pi\rho_0^2 a^5}{315\varepsilon_0}$$

2. The figure shows the cross section of a long conducting cylinder with inner radius a and an outer radius b. The cylinder carries a current out of the page. The current density in the cross section is given by $J = Ar$. What is the magnetic field $\vec{B}$ at a point that is at a distance r ($a < r < b$) from the central axis of the cylinder? (2 marks)



(4 marks)

Solution

Due to the symmetry of the problem the generated magnetic field is tangent to the Amperian loop. Applying Ampere's law we get:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

$$B2\pi r = \mu_0 \int \mathbf{J} \cdot d\mathbf{A}$$

$$B2\pi r = 2\pi\mu_0 \int_a^b Ar'r'dr'$$

$$B = \frac{A\mu_0}{r} \int_a^r r'^2 dr'$$

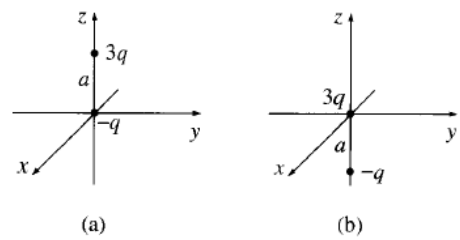
$$B = \frac{A\mu_0}{3r}(r^3 - a^3)$$

$$\mathbf{B} = \frac{A\mu_0}{3r}(r^3 - a^3)\hat{\phi}$$

Section B: Please answer only ONE question

SECTION B (Solve ONLY one of the two following problems)

3. Two point charges, $3q$ and $-q$ by a distance a. For each of the arrangements shown in the figure find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential (in spherical coordinates) at large r (include both monopole and dipole contributions). (Hint: read carefully pages 149 and 150 of our textbook)



(4 marks)

Solution:

(a) (i) the monopole term is the total charge so $Q = 2q$. (ii) The dipole moment is given by: $\mathbf{p} = \sum_{i=1}^{2} q_i \mathbf{r}_i^{'} = -q \cdot 0 + 3qa\hat{\mathbf{k}} = 3qa\hat{\mathbf{k}}$ (iii) The approximate potential at large distances is given by the potential of the monopole term and the potential due to the dipole term:

$$V \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right) = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{\left(3qa\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{r}}}{r^2}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]$$

(b) (i) the monopole term is the total charge so $Q = 2q$. (ii) The dipole moment is given by: $\mathbf{p} = \sum_{i=1}^{2} q_i \mathbf{r}_i' = -q \cdot (-a\hat{\mathbf{k}}) = qa\hat{\mathbf{k}}$ (iii) The approximate potential at large distances is given by the potential of the monopole term and the potential due to the dipole term:

$$V \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right) = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{(qa\hat{\mathbf{k}})\cdot\hat{\mathbf{r}}}{r^2}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]$$

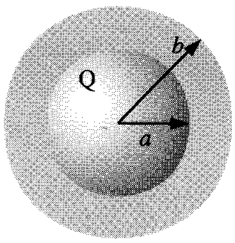
4. A metal sphere of radius a carries a charge Q. It is surrounded, out to a radius b, by a linear dielectric material of permittivity ε and electric susceptibility χ_e.

A) Find the electric field in all regions of space. (Inside the conductor, in the dielectric and in the vacuum) (1 mark).

B) Find the potential at the center of the sphere (relative to infinity) (1 mark).

C) Find the polarization of the dielectric (1 mark).

D) Find the surface bound charge densities in the inner and outer surfaces of the dielectric (1 mark).



Solution:

A) Inside the conductor there is no electric field. Thus **E** = **D** = **P** = **0**.

Also the free charge in the system is Q. Thus applying Gauss' law for the electric displacement **D** or a Gaussian surface of radius r inside the dielectric (i.e. $a < r < b$) we get:

$$\int \mathbf{D}\cdot d\mathbf{S} = Q \Rightarrow \int D\hat{\mathbf{r}}\cdot(dS\hat{\mathbf{r}}) = Q \Rightarrow$$

$$D\int dS = Q \Rightarrow D4\pi r^2 = Q$$

$$D = \frac{Q}{4\pi r^2}$$

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}$$

The electric field is related to the displacement by **E** = **D**/ε. Thus the electric field is given by:

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}}, & a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > b \end{cases}$$

B) Thus for the potential we have:

$$V(r) = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^0 (E\hat{\mathbf{r}})\cdot(dr'\hat{\mathbf{r}}) = -\int_\infty^0 Edr$$

but the integral is split into three parts:

$$V(r) = -\int_\infty^0 Edr = -\int_\infty^b Edr - \int_b^a Edr - \int_a^0 Edr$$

$$V(r) = -\int_\infty^b \frac{Q}{4\pi\varepsilon_0 r^2}dr - \int_b^a \frac{Q}{4\pi\varepsilon r^2}dr - \int_a^0 0dr$$

$$V(r) = \frac{Q}{4\pi}\left\{-\frac{1}{\varepsilon_0}\int_\infty^b \frac{1}{r^2}dr - \frac{1}{\varepsilon}\int_b^a \frac{1}{r^2}dr\right\}$$

$$V(r) = \frac{Q}{4\pi}\left\{\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right\}$$

C) The polarization of the material is given by:

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} = \frac{\varepsilon_0\chi_e Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}}$$

D) The surface bound charge density is given by:

$$\sigma_\mathrm{b} = \mathbf{P}\cdot\hat{\mathbf{n}}|_\mathrm{surface} = \begin{cases} \mathbf{P}\cdot\hat{\mathbf{n}}|_\text{outer surface} \\ \mathbf{P}\cdot\hat{\mathbf{n}}|_\text{inner surface} \end{cases}$$

$$\sigma_\mathrm{b} = \begin{cases} \mathbf{P}\cdot\hat{\mathbf{r}}|_\text{outer surface} \\ -\mathbf{P}\cdot\hat{\mathbf{r}}|_\text{inner surface} \end{cases}$$

$$\sigma_\mathrm{b} = \begin{cases} \dfrac{\varepsilon_0\chi_e Q}{4\pi\varepsilon b^2}, & \text{(outer surface)} \\ -\dfrac{\varepsilon_0\chi_e Q}{4\pi\varepsilon a^2} & \text{(inner surface)} \end{cases}$$