PHYS 501
2nd Midterm Exam - FALL 2019
Tuesday 19th November 2019

Instructor: Prof. V. Lempesis
Duration: 3 hours

Please answer all questions

1. A conducting wire along the z-axis carries a current *I*. The resulting magnetic vector potential in cylindrical coordinates is given by $\mathbf{A} = \hat{\mathbf{k}}\frac{\mu I}{2\pi}\ln\left(\frac{1}{\rho}\right)$. Show the magnetic induction $\mathbf{B} = \vec{\boldsymbol{\nabla}}\times\mathbf{A}$ is given by $\mathbf{B} = \hat{\varphi}_0\frac{\mu I}{2\pi\rho}$.

(5 marks)

Solution:

$$\mathbf{B} = \vec{\nabla}\times\mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}}_0 & \rho\hat{\phi}_0 & \hat{\mathbf{k}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{vmatrix} =$$

$$\frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}}_0 & \rho\hat{\phi}_0 & \hat{\mathbf{k}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu_0 I}{2\pi}ln\left(\frac{1}{\rho}\right) \end{vmatrix} =$$

$$\frac{1}{\rho}\left\{-\rho\hat{\phi}_0\left[\frac{\partial}{\partial\rho}\left(\frac{\mu_0 I}{2\pi}ln\left(\frac{1}{\rho}\right)\right)\right]\right\} = -\hat{\phi}_0\frac{\mu_0 I}{2\pi}\frac{\partial}{\partial\rho}ln\left(\frac{1}{\rho}\right) =$$

$$-\frac{\mu_0 I}{2\pi}\frac{1}{\left(\frac{1}{\rho}\right)}\left(-\frac{1}{\rho^2}\right)\hat{\phi}_0 =$$

$$\frac{\mu_0 I}{2\pi\rho}\hat{\phi}_0$$

2. A certain force field is given (in spherical polar coordinates) by

$$\mathbf{F} = \hat{\mathbf{r}}_0\frac{2P\cos\theta}{r^3} + \hat{\theta}_0\frac{P}{r^3}\sin\theta, \qquad r \geq P/2$$

Calculate $\vec{\nabla}\times\mathbf{F}$.

(5 marks)

Solution:

$$\vec{\nabla}\times F=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{r}}_0 & r\hat{\theta}_0 & r\sin\theta\hat{\phi}_0\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{r}}_0 & r\hat{\theta}_0 & r\sin\theta\hat{\phi}_0\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ 2P\cos\theta/r^3 & P\sin\theta/r^2 & 0\end{vmatrix}$$

$$\frac{1}{r^2\sin\theta}\left\{\hat{\mathbf{r}}_0\left[\frac{\partial 0}{\partial\theta}-\frac{\partial}{\partial\phi}\left(\frac{P\sin\theta}{r^2}\right)\right]-r\hat{\theta}_0\left[\frac{\partial}{\partial r}(0)-\frac{\partial}{\partial\phi}\left(\frac{2P\cos\theta}{r^3}\right)\right]\right.$$
$$\left.+r\sin\theta\hat{\phi}_0\left[\frac{\partial}{\partial r}\left(\frac{P\sin\theta}{r^2}\right)-\frac{\partial}{\partial\theta}\left(\frac{2P\cos\theta}{r^3}\right)\right]\right\}$$

$$\frac{1}{r^2\sin\theta}\left\{Pr\sin\theta\hat{\phi}_0\left[-\frac{2\sin\theta}{r^3}+2\frac{\sin\theta}{r^3}\right]\right\}=0$$

3. Working in cylindrical coordinates we know for the unit vectors the following relations:

$$\hat{\rho}_0=\mathbf{i}\cos\phi+\mathbf{j}\sin\phi,\quad \hat{\phi}_0=-\mathbf{i}\sin\phi+\mathbf{j}\cos\phi,\quad \hat{\mathbf{k}}=\hat{\mathbf{k}}$$

Calculate the quantities: $\frac{\partial\hat{\rho}_0}{\partial\phi}$, $\frac{\partial\hat{\phi}_0}{\partial\phi}$, $\frac{\partial\hat{\rho}_0}{\partial\rho}$.

Solution:

$$\frac{\partial\hat{\rho}_0}{\partial\phi}=\frac{\partial}{\partial\phi}\left(\mathbf{i}\cos\phi+\mathbf{j}\sin\phi\right)=\mathbf{i}\frac{\partial\cos\phi}{\partial\phi}+\mathbf{j}\frac{\partial\sin\phi}{\partial\phi}=-\mathbf{i}\sin\phi+\mathbf{j}\cos\phi=\hat{\phi}_0$$

$$\frac{\partial\hat{\phi}_0}{\partial\phi}=\frac{\partial}{\partial\phi}\left(-\mathbf{i}\sin\phi+\mathbf{j}\cos\phi\right)=-\mathbf{i}\frac{\partial\sin\phi}{\partial\phi}+\mathbf{j}\frac{\partial\cos\phi}{\partial\phi}=-\mathbf{i}\cos\phi-\mathbf{j}\sin\phi=-\hat{\rho}_0$$

$$\frac{\partial \hat{\rho}_0}{\partial \rho} = 0$$

4. Evaluate the integral $\oint \mathbf{r} \cdot d\mathbf{r}$ by Stoke's theorem.

Solution:

$$\oint \mathbf{V} \cdot d\mathbf{r} = \int_S (\vec{\nabla} \times \mathbf{V}) \cdot (\mathbf{n} dA)$$

Let **V** = **r** then we have

$$\oint \mathbf{r} \cdot d\mathbf{r} = \int_S (\vec{\nabla} \times \mathbf{r}) \cdot (\mathbf{n} dA)$$

but

$$\vec{\nabla} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} =$$

$$\mathbf{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

So

$$\oint \mathbf{r} \cdot d\mathbf{r} = 0$$

Mathematical Supplement:

$$\vec{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\mathbf{i} + \frac{\partial \Phi}{\partial y}\mathbf{j} + \frac{\partial \Phi}{\partial z}\mathbf{k}, \quad \vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{\nabla} \times \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}}_0 & \rho\hat{\phi}_0 & \hat{\mathbf{k}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & F_\phi & F_z \end{vmatrix}$$

$$\nabla^2 f(x, y, z) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Mathematical Supplement:

$$\vec{\nabla}\cdot\mathbf{V}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho V_{\rho}\right)+\frac{1}{\rho}\frac{\partial V_{\varphi}}{\partial\varphi}+\frac{\partial V_{z}}{\partial z},\qquad \vec{\nabla}\times\mathbf{V}=\frac{1}{\rho}\begin{vmatrix}\hat{\rho}_0 & \rho\hat{\varphi}_0 & \mathbf{k}\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z}\\ V_{\rho} & \rho V_{\varphi} & V_{z}\end{vmatrix}$$

$$\oint \mathbf{V}\cdot d\mathbf{r}=\int_{S}\left(\vec{\nabla}\times\mathbf{V}\right)\cdot(\mathbf{n}dA)$$

$$\vec{\nabla}\times\mathbf{V}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{r}}_0 & r\hat{\theta}_0 & r\sin\theta\hat{\phi}_0\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ V_{r} & rV_{\theta} & r\sin\theta V_{\phi}\end{vmatrix}$$