PHYS 404
HANDOUT 13 - *Fourier Transform*

1. Prove the time and frequency shifting properties of the Fourier transform.

(Sch. p. 77)

2. Prove the so called *modulation theorem* which states:

$$F\left[g\left(t\right)\cos\left(\omega_0 t\right)\right] = \frac{1}{2}G\left(\omega - \omega_0\right) + \frac{1}{2}G\left(\omega + \omega_0\right)$$

where $F\left[f\left(t\right)\right] = G\left(\omega\right)$.

(Sch. p. 78)

3. Study the Fourier transform of a *real* function $g(t)$.

(Sch. p. 79)

4. Show that if the Fourier transform of a real function $g(t)$ is real, then $g(t)$ is an even function of t, and if the Fourier transform of a real function $g(t)$ is purely imaginary, then $g(t)$ is an odd function of t.

(Sch. p. 89)

5. Show that if $g(t)$ is a periodic function with period T then its Fourier transform is given by

$$G(\omega) = 2\pi \sum_{n=-\omega}^{\infty} c_n \delta\left(\omega - n\omega_0\right)$$

Where the coefficients c_n are associated with the corresponding Fourier series.

(Sch. p. 104)

6. Find the Fourier transform of $\delta'(t)$.

(Sch. p. 108)

7. Find the Fourier transform of $\sin^3 t$.

(Sch. p. 113)

8. Find the Fourier transform of the function

$$f(t) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad \text{period } 2\pi .$$

9. From the expression for Fourier transform, show what happens if we consider the substitution :

$$\begin{cases} t \to \ln x \\ i\omega \to \alpha - \gamma \end{cases}$$

(Arf. p. 797)

10. The finite wave-train is produced when an infinite wave train of the form $\sin\omega_0 t$ is clipped by Kerr cell or saturable dye cell shutters so that we have:

$$g(t) = \begin{cases} \sin\omega_0 t, & |t| < N\pi/\omega_0 \\ 0, & |t| > N\pi/\omega_0 \end{cases}$$

(Arf. p. 801)

11. Find the Fourier transform of the pulse:

$$g(t) = \begin{cases} A/T, & -T < t < 0 \\ -A/T, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

(Sch. p. 91)

12. Find the Fourier transform of the triangular pulse:

$$g(x) = \begin{cases} h\left(1 - a|x|\right), & |x| < 1/a, \\ 0, & |x| > 1/a. \end{cases}$$

(Arf. p. 804)

13. In a resonant cavity an electromagnetic oscillation of frequency ω_0 dies out as:

$$A(t) = A_0 e^{-\omega_0 t/2Q} e^{-i\omega_0 t}, \quad t > 0$$

(Take $A(t) = 0$ for $t < 0$.)
This parameter Q is a measure of the ratio of stored energy to energy loss per cycle. Calculate the frequency distribution of the oscillation, $a^*(\omega)a(\omega)$, where $a(\omega)$ is the Fourier transform of $A(t)$.

(Arf. p. 805)

14. Prove that

$$\frac{\hbar}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}d\omega}{E_0 - i\Gamma/2 - \hbar\omega} = \begin{cases} \exp\left(-\Gamma t/2\hbar\right)\exp\left(-iE_0 t/\hbar\right), & t > 0 \\ 0, & t < 0 \end{cases}$$

(Sch. p. 16)

15. A rectangular pulse is described by

$$g(t) = \begin{cases} 1, & |t| < d/2 \\ 0, & |t| > d/2 \end{cases}.$$

Show that the Fourier exponential transform is

$$G(\omega) = d\frac{\sin\left(\omega d/2\right)}{\left(\omega d/2\right)}.$$

Here is the single slit diffraction problem of physical optics. The slit is described by $g(t)$. The diffraction pattern *amplitude* is given by the Fourier transform $G(\omega)$.

(Sch. p. 85)

16. Find the Fourier transform of the function $f(t) = e^{-a|t|}$ $(a > 0)$;

(Sch. p. 86)

17. The hydrogen atom ground state may be described by the special wave function

$$\psi\left(\mathbf{r}\right) = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}$$

where a_0 being the Bohr radius $\hbar/me^2$. Find the momentum wave function. *Hint: you will need the integral* $\int e^{-ar+i\mathbf{b}\cdot\mathbf{r}}d^3r = 8\pi a/\left(a^2+b^2\right)^2$.

(Arf. p. 816)

18. A free particle in quantum mechanics is described by the function

$$\psi_p\left(x,t\right) = \left(1/\sqrt{2\pi\hbar}\right)e^{i\left[px/\hbar-\left(p^2/2m\hbar\right)t\right]}$$

Combining waves of adjacent momentum with an amplitude weighting factor $c(p)$, we form a wavepacket

$$\Psi\left(x,t\right) = \int_{-\infty}^{\infty} c\left(p\right)\psi_p\left(x,t\right)dp.$$

(a) Solve for $c\left(p\right)$ given that $\Psi\left(x,0\right) = \sqrt[4]{\frac{\lambda}{\pi}}e^{-\lambda x^2/2}$.

(b) Using the known value of $c(p)$, integrate to get the explicit form of $\Psi\left(x,t\right)$. Note that this wave packet diffuses or spreads out with time.

Hint: You will need the integral: $\int_{-\infty}^{\infty}\exp(-a^2x^2 \pm bx)dx = \exp\left(\frac{b^2}{4a^2}\right)\frac{\sqrt{\pi}}{a}$.

Note: An interesting discussion of this problem from the evolution operator point of view is given by S. M. Blinder, "Ecolution of a Gaussian wavepacket." Am. J. Phys. ***36****, 525 (1968)*

(Arf. p. 819)

19. The one dimensional time-independent Schroedinger wave equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x)$$

For the special case of $V(x)$ an analytic function of *x,* show that the corresponding momentum wavefunction is

$$V\left(i\hbar\frac{d}{dp}\right)g(p)+\frac{p^2}{2m}g(p)=Eg(p).$$

Where $V\left(i\hbar\frac{d}{dp}\right)$ is the potential function with argument $(i\hbar d/dp)$.

(a) Derive this equation directly from the Fourier transform.

(b) Find the Schroedinger equation for the simple haromic oscillation in the momentum representation.

(Arf. p. 820)

20. The nuclear form factor $F(k)$ and the charge distribution $\rho(r)$ are three-dimensional Fourier transforms of each other:

$$F(k)=\frac{1}{(2\pi)^{3/2}}\int\rho(r)e^{i\mathbf{k}\cdot\mathbf{r}}d^3r.$$

If the measured form factor is

$$F(k)=(2\pi)^{-3/2}\left(1+\frac{k^2}{a^2}\right)^{-1},$$

find the corresponding charge distribution.

(Arf. p. 819)

21. The ordinary space wave function $\psi(\mathbf{r},t)$ satisfies the time-dependent Schroedinger equation

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V(\mathbf{r})\psi.$$

Show that the corresponding time-dependent momentum function satisfies the analogous equation

$$i\hbar\frac{\partial\varphi(\mathbf{p},t)}{\partial t}=\mathbf{p}^2\varphi+V(i\hbar\nabla_p)\varphi.$$

Hint: $V\left(i\hbar\nabla_p\right)$ *is the same function of the variable* $i\hbar\nabla_p$ *that* $V(\mathbf{r})$ *is of the variable* $\mathbf{r}$.

22. The *n*th moment m_n of a function is defined by

$$m_n = \int_{-\infty}^{\infty} t^n g(t)dt \qquad n = 0,\ 1,\ 2,...$$

Show that

$$m_n = i^n \frac{d^n G(0)}{d\omega^n} \qquad n = 0,\ 1,\ 2,...$$

where

$$\frac{d^n G(0)}{d\omega^n} = \left.\frac{d^n G(\omega)}{d\omega^n}\right|_{\omega=0} \quad \text{and} \quad G(\omega) = F\left[g(t)\right]$$

(Sch. p. 95)

23. Use the result of the previous problem to show that

$$G(\omega) = \sum_{n=}^{\infty} \left(-i\right)^n m_n \frac{\omega^n}{n!}$$

(Sch. 95)