PHYSICS 301

Lecture 10

Applications of Residues

The Cauchy Principal Value

- In this lecture we are going to consider methods of calculation of integrals of a real function $f(x)$ of the form:

$$I = \int_{-\infty}^{+\infty} f(x)dx$$

- We say that such an integrals **converges** if the two limits in the relation:

$$I = \lim_{L\to\infty} \int_{-L}^{\alpha} f(x)dx + \lim_{\to\infty} \int_{\alpha}^{R} f(x)dx\,, with\ \alpha\ finite$$

they **do exist**.

The Cauchy Principal Value

- When we calculate integrals, in complex analysis, it is useful (as we will see) to consider a more limited condition letting $L = R$. The limit which we get in this case is called **Cauchy principal value**.

$$I_p = \lim_{R \to \infty} \int_{-R}^{R} f(x) dx$$

- If the relation for I converges then $I = I_p$ by letting simply $L = R$ as a specific case.
- It is possible I_p to exist, while I not. For example when $f(x)$ is odd $I_p = 0$ but I does not exist.

The Cauchy Principal Value

- In applications normally we check first the convergence of I using the usual calculus criteria and **then** we calculate the integral with the help of the I_p.
- **As a first step** we are going to discuss how we can calculate integrals of the form:

$$I = \int_{-\infty}^{+\infty} f(x)dx\,,$$

for $f(x) = N(x)/D(x)$, where $N(x), D(x)$ are real polynomials, i.e. $f(x)$ is a rational faction. Also $D(x) \neq 0$ for any $x \in \mathbb{R}$ and its degree larger by at least 2 than the degree of $N(x)$, so $f(x)$ definitely converges. The method has as follows:

The Cauchy Principal Value

- We conisder the integral:

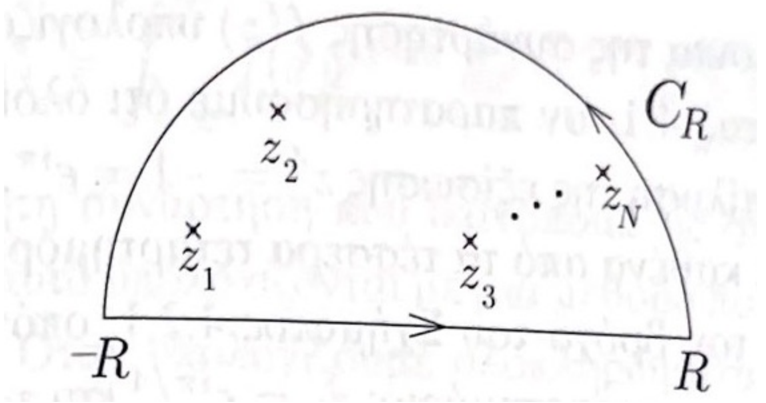
$$I = \oint_C f(z)dz = \int_{-R}^{R} f(x)dx + \oint_{C_R} f(z)dz\,,$$

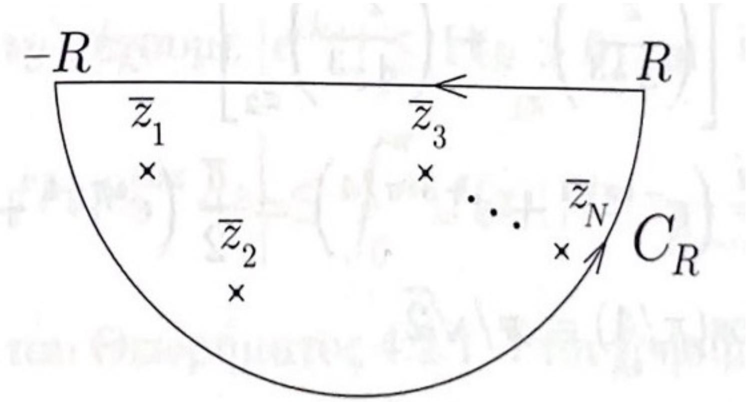
Where C_R is a large semicircle and the loop C contains all the singularities of $f(z)$, i.e. all the points $z_1, z_2, \ldots, z_n$ for which $D(z) = 0$.

- We use Cauchy's residue theorem and we show that $\lim_{R\to\infty} \oint_{C_R} f(z)dz = 0$, so at the limit $R \to \infty$ we get:

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum_{j=1}^{N} \mathrm{Res}\big(f(z); z_j\big)$$

- Note that since $D(x)$ is a real polynomial its complex roots make up couples of conjugate numbers.





Improper Integrals from Fourier Analysis

The residue theory can be applied in the evaluation of convergent improper integrals of the form:

$$\int_{-\infty}^{\infty} f(x)\sin ax\, dx \;\; or \int_{-\infty}^{\infty} f(x)\cos ax\, dx\,,$$

Where a denotes a positive constant. We assume $f(x) = p(x)/q(x)$ where $p(x), q(x)$ are polynomials with real coefficients and no factors in common. Also $q(z)$ has no real zeros.

Jordan's Lemma

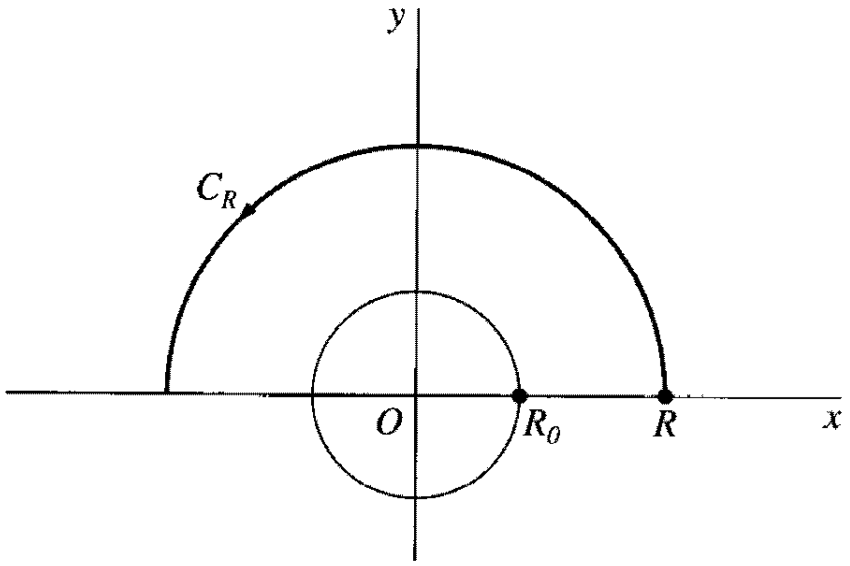
In the evaluation of integrals of the type treated in previous slide, it is sometimes necessary to use Jordan's lemma, which is stated here as a theorem :

Suppose that:

1. A function $f(z)$ is analytic at all points z in the upper half plane $y \geq 0$ that are exterior to a circle $|z| = R_0$.
2. C_R denotes a semicircle $z = Re^{i\theta}$, $(0 \leq \theta \leq \pi)$, where $R > R_0$Where a denotes a positive constant.
3. For all points z on C_R, there is a positive constant M_R such that $|f(z)| < M_R$, where $\lim_{R\to\infty} M_R = 0$.

Then for every positive constant a,

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0$$



Indented Paths

Theorem: Suppose that:

1. A function $f(z)$ has a simple pole at a point $z = x_0$ on the real axis, with a Laurent series representation in a punctured disk $0 < |z - x_0| < R_2$ and with residue B_0;
2. C_ρ denotes the upper half of a circle $|z - x_0| = \rho$, where $\rho < R_2$ and the clockwise direction is taken, then

Then for every positive constant a,

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)\, dz = -B_0 \pi i$$