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On a Singular Non local Fractional System Describing a Generalized Timoshenko System with Two Frictional Damping Terms

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Abstract: This paper concerns a nonhomogeneous singular fractional order system, with two frictional damping terms. This system can be considered as a generalization of the so-called Timoshenko system. Results on the existence, uniqueness, and continuous dependence on the solution were obtained via an energy approach, which mainly relies on a priori bounds and density arguments. The approach relies on functional analysis tools and operator theory. Very few results concerning the well-posedness of fractional order Timoshenko systems can be found in the literature. Our results generalize and improve the previous ones and significantly boost the development of the used method.

Keywords: fractional system; frictional damping; nonlocal; a priori bound; nonlocal condition; singular system

MSC: 35B45; 35R11; 35L55

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1. Introduction

The field of structural mechanics has witnessed significant advancements over the past century, with researchers and engineers continually striving to develop more accurate and efficient models for analyzing the behavior of elastic structures. One such advancement is the so-called Timoshenko system [1]

$$\begin{cases} \rho_1 Y_{tt} - \kappa(Y_x - P)_x = 0, & (x,t) \in (0,L) \times (0,\infty) \\ \rho_2 P_{tt} = \kappa^* P_{xx} + \kappa(Y_x - P) & (x,t) \in (0,L) \times (0,\infty), \\ (Y_x - P)\mid_{x=0}^{x=L} = 0, & P_x \mid_{x=0}^{x=L} = 0, \end{cases} \tag{1}$$

describing the transverse vibration of a beam of length L in its equilibrium position, with a transverse displacement Y and a rotation angle P. This system offers a comprehensive description of the dynamic behavior of elastic structures, including beams, plates, and shells. The Timoshenko system has found broad applications in various engineering disciplines, such as civil, mechanical, and aerospace engineering, due to its ability to more accurately represent the physical behavior of elastic structures compared to classical models like the Euler–Bernoulli beam Equation (see [2–7]).

The motivation for studying the capacity of Timoshenko systems is to incorporate both the translational and rotational motion of elastic structures, providing a more detailed understanding of their dynamic response, stability, and vibration characteristics. As modern engineering applications demand the use of lightweight materials and efficient designs, there is a growing need for advanced mathematical tools to analyze the performance of elastic structures under a wide range of loading conditions and environments.

While significant progress has been made in understanding the existence, uniqueness, stability (decay, blow up) and controllability of solutions for Timoshenko systems [8–18], several research gaps remain that warrant further investigation. Some of these gaps include:

- Nonlinearities: The majority of previous studies have focused on linear Timoshenko systems, while real-world applications often involve nonlinear damping, source terms, or material properties. Investigating the behavior of Timoshenko systems in the presence of various types of nonlinearities is crucial for better understanding the structural response under complex loading conditions.
- Coupled problems: The interaction between different physical phenomena, such as thermal effects, fluid–structure interactions, or piezoelectric coupling, has not been extensively studied in the context of Timoshenko systems. Developing mathematical models that account for these coupled effects is essential for accurately predicting the behavior of advanced materials and structures.
- Time-dependent coefficients and boundary conditions: Most existing research assumes time-invariant coefficients and boundary conditions, which may not accurately represent real-world scenarios where material properties or constraints change over time. Exploring the existence and uniqueness of solutions for Timoshenko systems with time-dependent coefficients and boundary conditions is an important area for future research.
- Numerical methods and computational efficiency: While various numerical methods have been proposed for solving Timoshenko systems, there is still room for improvement in terms of computational efficiency and accuracy, particularly for large-scale problems and high-performance computing applications. We can also mention some related works on fractional-order dynamical models such as [19–21]. To the best of our knowledge, the investigation of the fractional system problem (2)–(4) has never been explored in the literature. Most of papers in the literature dealing with Timoshenko systems are related to classical Timoshenko systems with classical boundary conditions. Our system is a generalization of the classical Timoshenko system of type (1), where our considered system is singular and the associated boundary conditions are nonlocal.The obtained results on the well-posedness of the proposed fractional problem in this present article can be viewed and considered as a contribution to the development of the energy inequality method, which is mainly used to prove the well-posedness of mixed problems with integral boundary conditions.

Motivated by the above results on Timoshenko systems, we consider a nonlocal problem for a non-homogeneous fractional Timoshenko system with a frictional damping in both equations. The system is associated with initial conditions and Neumann and integral conditions of the form $\int_0^l \mu(x)u(x,t)dx = f(t)$, where $\mu(x)$ and $f(t)$ are given functions. This integral condition may represent a mean, a total flux, and a total energy. It can appear in several fields of engineering, such as different vibration problems, heat diffusion, thermoelasticity, and plasma physics. The reader can refer to [22–36]. The presence of the integral condition creates great difficulties while performing different integrations and calculations, particularly for the fractional cases as in our problem. For fractional boundary initial value problems with nonlocal conditions, the reader can refer to [37–44].

2. The Problem Setting and Functional Spaces Frame

Let us consider a non-homogeneous singular fractional beam model (generalized Timoshenko system) with two frictional damping terms, in the domain $Q = I \times [0,T]$ with $I = [0,L]$.

$$\begin{cases} \mathcal{L}_1(Y,P) = \rho_1 \partial_t^{\alpha+1} Y - \kappa_1 \frac{1}{x}(xY_x)_x - \kappa_1 P_x + Y_t = F(x,t) \\ \mathcal{L}_2(Y,P) = \rho_2 \partial_t^{\alpha+1} P - \kappa_2 \frac{1}{x}(xP_x)_x + \kappa_1 (Y_x + P) + P_t = G(x,t), \end{cases} \tag{2}$$

The fractional operator $\partial_t^{\alpha+1}$of order $\alpha+1$, with $0<\alpha<1$, is taken in the Caputo sense defined by (13).

With (2), we associate the following initial conditions

$$\begin{cases} l_1 Y = Y(x,0) = \varphi(x), \ l_2 Y = Y_t(x,0) = \psi(x)\,, \\ l_1 P = P(x,0) = f(x), \ l_2 P = P_t(x,0) = g(x), \end{cases} \tag{3}$$

and the boundary classical and integral conditions

$$\int_0^L xYdx = 0,\, Y_x(L,t) = 0,\ \int_0^L xPdx = 0,\, P_x(L,t) = 0. \tag{4}$$

The constants ρ_1, ρ_2, κ_1, and κ_2 are positive, and f, g, φ, ψ, F, and G are given functions which will be specified later on.

We suppose that the input data satisfy the compatibility conditions

$$\int_0^L x\varphi dx = 0,\, \varphi_x(L,t) = 0,\ \int_0^L x\psi dx = 0,\, \psi_x(L,t) = 0. \tag{5}$$

Our main goal is to investigate the well-posedness of the problem (2)–(4) by using the energy method inspired from functional analysis (tools based on operator theory).

Let us first introduce the functional space frame of the problem. Let $L^2_\rho(Q)$ be the weighted Hilbert space of square integrable functions on the domain $Q^T = (0,L)\times(0,T)$, $T<\infty$, with inner product

$$(u,v)_{L^2_\rho(Q^T)} = \int_{Q^T} xuvdxdt. \tag{6}$$

Let $B^1_2(0,L)$ [31] be the Hilbert space of square summable primitive functions, which can be considered as a completion of space $C_0(0,L)$ for the scalar product

$$(u,v)_{B^1_2(0,L)} = \int_0^L \Im_x u.\Im_x v dx,$$

where $\Im_x u = \int_0^x u(\zeta)d\zeta, \forall\, x\in(0,L)$. The associated norm is $\|u\|^2_{B^1_2(0,L)} = \sqrt{(u,u)_{B^1_2(0,L)}} = \int_0^L (\Im_x u)^2 dx$. We denote by $C(I;L^2(0,L))$ with $I=(0,T)$ the space of continuous functions $\theta(.,t): I\to L^2(0,L)$ with norm

$$\|\theta\|^2_{C(I;L^2(0,L))} = \sup_{0\le t\le T}\|\theta(.,t)\|^2_{L^2(0,L)} < \infty, \tag{7}$$

and let $C(\overline{I};B^1_2(0,L))$ be the set of functions $\theta(.,t):\overline{I}\to B^1_2(0,L)$ with norm

$$\|\theta\|^2_{C(\overline{I};B^1_2(0,L))} = \sup_{0\le t\le T}\|\Im_x\theta(.,t)\|^2_{L^2(0,L)} = \sup_{0\le t\le T}\|\theta(.,t)\|^2_{B^1_2(0,L)} < \infty. \tag{8}$$

We also introduce the weighted Hilbert space $L^2(I;H^1_\rho(0,L))$ with $I=(0,T)$ as the space of summable functions $u(.,t): I\to H^1_\rho(0,L)$ with the norm

$$\|u\|^2_{L^2(I;H^1_\rho(0,L))} = \|u\|^2_{L^2(I;L^2_\rho(0,L)} + \|u_x\|^2_{L^2(I;L^2_\rho(0,L)}. \tag{9}$$

The problem (2)–(4) can be written in the operator form: $\Sigma\mathcal{F} = H$ with $\mathcal{F} = (Y, P)$, $\Sigma\mathcal{F} = (\mathcal{F}_1(Y, P), \mathcal{F}_2(Y, P))$, and $H = (H_1, H_2)$, where

$$\begin{cases} \mathcal{F}_1(Y,P) = \{\mathcal{L}_1(Y,P), l_1Y, l_2Y\} \\ \mathcal{F}_2(Y,P) = \{\mathcal{L}_2(Y,P), l_1P, l_2P\} \\ H_1 = \{F, \simeq, \psi)\},\ H_2 = \{G, f, g\}. \end{cases}, \tag{10}$$

We denote by $D(\Sigma)$ the domain of the operator Σ, consisting of elements $(Y, P) \in \left(L^2(\bar{I}; L^2(0, L))\right)^2$ such that $Y_x, P_x, Y_t, P_t, Y_{tt}, P_{tt}, Y_{xx}, P_{xx}$, $\partial_t^{\alpha+1}Y$, $\partial_t^{\alpha+1}P$ belong to $L^2(\bar{I}; L^2(0, L))$, verifying the initial and boundary conditions (3) and (4). The operator $\Sigma : B \to E$, where B is the Banach space with respect to the norm

$$\begin{aligned} \|\mathcal{F}\|_B^2 \;=\; & \|Y\|^2_{L^2(0.,T,H^1_\rho(0,L))} + \|P\|^2_{L^2(0.,T,H^1_\rho(0,L))} + \|Y\|^2_{C(0.,T,L^2_\rho(0,L))} \\ & + \|P\|^2_{C(0.,T,L^2_\rho(0,L))}. \end{aligned} \tag{11}$$

and $E = \left\{L^2(Q^T) \times L^2_\rho(Q^T) \times L^2(0, L)\right\} \times \left\{L^2(Q^T) \times L^2_\rho(Q^T) \times L^2(0, L)\right\}$ is the Hilbert space consisting of vector-valued functions $H = (\{F, \simeq, \psi)\}, \{G, f, g\})$ for which the norm

$$\begin{aligned} \|H\|_E^2 \;=\; & \|\varphi\|^2_{L^2_\rho(0,L)} + \|f\|^2_{L^2_\rho(0,L)} + \|F\|^2_{L^2(0.,t,L^2(0,L))} + \|G\|^2_{L^2(0.,t,L^2(0,L))} \\ & + \|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)}. \end{aligned} \tag{12}$$

is finite.

3. Preliminaries

This section is devoted to exhibiting some tools (definitions, lemmas) needed for our proofs.

Definition 1 ([45])**.** *The time fractional derivative of order ν in the Caputo sense with $1 < \nu < 2$ for a function h is defined by*

$$\partial_t^\nu h(x,t) = \frac{1}{\Gamma(2-\nu)} \int_0^t \frac{h_{\tau\tau}(x,\tau)}{(t-\tau)^{\nu-1}} d\tau, \tag{13}$$

and for $\nu \in (0, 1)$ the fractional derivative reads

$$\partial_t^\nu h(x,t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{h_{\tau}(x,\tau)}{(t-\tau)^{\nu}} d\tau \tag{14}$$

where $\Gamma(\nu) = \int_0^\infty t^{\nu-1}e^{-t}dt$ is the Gamma function.

Definition 2 ([45])**.** *The fractional Riemann–Liouville integral of order $0 < \nu < 1$ is given by*

$$D_t^{-\nu} h(x,t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{h(x,\tau)}{(t-\tau)^{1-\nu}} d\tau. \tag{15}$$

Lemma 1 ([46])**.** *Let $H(s) \geq 0$ be a function absolutely continuous on $[0, T]$, and assume that for $\forall\, s \in [0, T]$, H verifies*

$$\frac{dH}{ds} \leq m_1(s)H(s) + m_2(s), \tag{16}$$

where the functions $m_1(s)$ and $m_2(s)$ are summable and non-negative on $[0,T]$. Then

$$H(s) \leq \exp\left\{ \int_0^s m_1(t)dt \right\} \left(m_1(0) + \int_0^s m_2(t)dt \right). \tag{17}$$

Lemma 2 ([39]). *Let a non-negative absolutely continuous function $f(t)$ satisfy the inequality*

$$\partial_t^\beta f(t) \leq b_1 f(t) + b_2(t), \quad 0 < \beta < 1, \tag{18}$$

for all $t \in [0,T]$, where $b_1 > 0$ and $b_2(t)$ is an integrable non-negative function on $[0,T]$, then

$$f(t) \leq f(0) E_\beta(b_1 t^\beta) + \Gamma(\beta) E_{\beta,\beta}(b_1 t^\beta) D_t^{-\beta} b_2(t), \tag{19}$$

where

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)} \text{ and } E_{\beta,\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + \mu)}, \tag{20}$$

are the Mittag–Leffler functions.

We also need the following elementary inequality. For a function $f \in L^2(\Omega)$, we have

$$D_t^{-\alpha-1} \|f\|^2_{L^2(\Omega)} \leq \frac{t^\alpha}{\Gamma(\alpha+1)} \int_0^t \|f\|^2_{L^2(\Omega)} d\tau, \ 0 < \alpha < 1. \tag{21}$$

Lemma 3 ([39]). *For any absolutely continuous function $v(t)$ on $[0,T]$, the following inequality holds*

$$v(t)\, \partial_t^\alpha v(t) \geq \frac{1}{2} \partial_t^\alpha v^2(t), \quad 0 < \alpha < 1. \tag{22}$$

4. The a Priori Bound (Uniqueness of Solution)

In this section, we establish a priori bound for the solution from which we deduce the uniqueness and the continuous dependence of solution of problem (2)–(4) on the input data.

Theorem 1. *For any element $\mathcal{F} = (Y,P) \in D(\Sigma)$, the following a priori bound holds*

$$\begin{aligned} &\|\mathcal{F}\|_B^2 = \|Y\|^2_{L^2(0.,T,H^1_\rho(0,L))} + \|P\|^2_{L^2(0.,T,H^1_\rho(0,L))} + \|Y\|^2_{C(0.,T,L^2_\rho(0,L))} + \|P\|^2_{C(0.,T,L^2_\rho(0,L))} \\ \leq\ & \mathcal{C}^* \|\Sigma\mathcal{F}\|_E^2 = \mathcal{C}^* \Big(\|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} \\ &+ \|\varphi\|^2_{L^2_\rho(0,L)} + \|f\|^2_{L^2_\rho(0,L)} + \|F\|^2_{L^2(0.,T,L^2(0,L))} + \|G\|^2_{L^2(0.,T,L^2(0,L))} \Big), \end{aligned} \tag{23}$$

where $\mathcal{C}^*$ is a positive constant given by

$$\mathcal{C}^* = \mathcal{V} \max\left\{ 1, \frac{T^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)} \right\}, \tag{24}$$

with

$$\mathcal{V} = \Gamma(\alpha) E_{\alpha,\alpha}(\lambda T^\alpha) \left(\max\left\{ 1, \frac{T^\alpha}{\alpha\Gamma(\alpha)} \right\} \right), \text{ and } \lambda = Q^{**}(Q^{**} e^{Q^{**}T} + 1). \tag{25}$$

The constant Q^{**} is given by (62).

Proof. Define the integro-differential operators $\mathcal{O}_1 Y = -x\mathcal{I}_x^2(\xi Y_t)$, $\mathcal{O}_2 P = -x\mathcal{I}_x^2(\xi P_t)$, where

$$\mathcal{I}_x^2(\xi N) = \int_0^x \int_0^\xi \eta N(\eta, t) d\eta d\xi, \tag{26}$$

and consider the identity

$$\begin{aligned} &\left(\rho_1 \partial_t^{\alpha+1} Y, \mathcal{O}_1 Y\right)_{L^2(0,L)} - \kappa_1 (\tfrac{1}{x}(xY_x)_x, \mathcal{O}_1 Y)_{L^2(0,L)} - \kappa_1 (P_x, \mathcal{O}_1 Y)_{L^2(0,L)} \\ &+ (Y_t, \mathcal{O}_1 Y)_{L^2(0,L)} + (\rho_2 \partial_t^{\alpha+1} P, \mathcal{O}_2 P)_{L^2(0,L)} \\ &- \kappa_2 (\tfrac{1}{x}(xP_x)_x, \mathcal{O}_2 P)_{L^2(0,L)} + \kappa_1 ((Y_x + P), \mathcal{O}_2 P)_{L^2(0,L)} \\ + \quad &(P_t, \mathcal{O}_2 P)_{L^2(0,L)} \\ = \quad &(F(x,t), \mathcal{O}_1 Y)_{L^2(0,L)} + (G(x,t), \mathcal{O}_2 P)_{L^2(0,L)}. \end{aligned} \tag{27}$$

We also define the classical differential operators $\mathcal{O}_3 P = xP$ and $\mathcal{O}_4 Y = xY$ and consider the identity

$$\begin{aligned} &\left(\rho_1 \partial_t^{\alpha+1} Y, \mathcal{O}_4 Y\right)_{L^2(0,L)} - \kappa_1 (\tfrac{1}{x}(xY_x)_x, \mathcal{O}_4 Y)_{L^2(0,L)} - \kappa_1 (P_x, \mathcal{O}_4 Y)_{L^2(0,L)} \\ &+ (Y_t, \mathcal{O}_4 Y)_{L^2(0,L)} + \rho_2 (\partial_t^{\alpha+1} P, \mathcal{O}_3 P)_{L^2(0,L)} \\ &- \kappa_2 (\tfrac{1}{x}(xP_x)_x, \mathcal{O}_3 P)_{L^2(0,L)} + \kappa_1 ((Y_x + P), \mathcal{O}_3 P)_{L^2(0,L)} \\ &+ (P_t, \mathcal{O}_3 P)_{L^2(0,L)} \\ = \quad &(F(x,t), \mathcal{O}_4 Y)_{L^2(0,L)} + (G(x,t), \mathcal{O}_3 P)_{L^2(0,L)} \end{aligned} \tag{28}$$

The use of conditions (3) and (4) and evaluation of each term in (27) yield

$$\left(\rho_1 \partial_t^{\alpha+1} Y, \mathcal{O}_1 Y\right)_{L^2(0,L)} = \rho_1 (\partial_t^\alpha (\mathcal{I}_x(\xi Y_t)), \mathcal{I}_x(\xi Y_t))_{L^2(0,L)}, \tag{29}$$

$$\left(\rho_2 \partial_t^{\alpha+1} P, \mathcal{O}_2 P\right)_{L^2(0,L)} = \rho_2 (\partial_t^\alpha (\mathcal{I}_x(\xi P_t), \mathcal{I}_x(\xi P_t))_{L^2(0,L)}, \tag{30}$$

$$\kappa_1 (\frac{1}{x}(xY_x)_x, \mathcal{O}_1 Y)_{L^2(0,L)} = -\kappa_1 (Y_x, \mathcal{I}_x(\xi Y_t))_{L^2_\rho(0,L)}, \tag{31}$$

$$(Y_t, \mathcal{O}_1 Y)_{L^2(0,L)} = \|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)}, \tag{32}$$

$$-\kappa_2 \left(\frac{1}{x}(xP_x)_x, \mathcal{O}_2 P\right)_{L^2(0,L)} = -\kappa_2 (P_x, \mathcal{I}_x(\xi P_t))_{L^2_\rho(0,L)}, \tag{33}$$

$$-\kappa_1 (P_x, \mathcal{O}_1 Y)_{L^2(0,L)} = \kappa_1 (P_x, \mathcal{I}_x^2(\xi Y_t))_{L^2_\rho(0,L)}, \tag{34}$$

$$\kappa_1 (Y_x, \mathcal{O}_2 P)_{L^2(0,L)} = -\kappa_1 (Y_x, \mathcal{I}_x^2(\xi P_t))_{L^2_\rho(0,L)}, \tag{35}$$

$$\kappa_1 (P, \mathcal{O}_2 P)_{L^2(0,L)} = \frac{\kappa_1}{2} \frac{\partial}{\partial t} \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)}, \tag{36}$$

$$(P_t, \mathcal{O}_2 P)_{L^2(0,L)} = \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)}. \tag{37}$$

In the same fashion, the terms on the LHS of (28) can be evaluated as

$$\rho_1 \left(\partial_t^{\alpha+1} Y, \mathcal{O}_4 Y\right)_{L^2(0,L)} = \rho_1 \left(\partial_t^{\alpha+1} Y, Y\right)_{L^2_\rho(0,L)}, \tag{38}$$

$$-\kappa_1 (\frac{1}{x}(xY_x)_x, \mathcal{O}_4 Y)_{L^2(0,L)} = \kappa_1 \|Y_x\|^2_{L^2_\rho(0,L)}, \tag{39}$$

$$-\kappa_1(P_x, \mathcal{O}_4 Y)_{L^2(0,L)} = -\kappa_1(P_x, Y)_{L^2_\rho(0,L)}, \tag{40}$$

$$(Y_t, \mathcal{O}_4 Y)_{L^2(0,L)} = \frac{1}{2}\frac{\partial}{\partial t}\|Y\|^2_{L^2_\rho(0,L)}, \tag{41}$$

$$\rho_2(\partial_t^{\alpha+1} P, \mathcal{O}_3 P)_{L^2(0,L)} = \rho_2(\partial_t^{\alpha+1} P, P)_{L^2_\rho(0,L)}, \tag{42}$$

$$-\kappa_2(\frac{1}{x}(xP_x)_x, \mathcal{O}_3 P)_{L^2(0,L)} = \kappa_2\|P_x\|^2_{L^2_\rho(0,L)}, \tag{43}$$

$$\kappa_1(Y_x, \mathcal{O}_3 P)_{L^2(0,L)} = \kappa_1(Y_x, P)_{L^2_\rho(0,L)}, \tag{44}$$

$$\kappa_1((P), \mathcal{O}_3 P)_{L^2(0,L)} = \kappa_1\|P\|^2_{L^2_\rho(0,L)}. \tag{45}$$

$$(P_t, \mathcal{O}_3 P)_{L^2(0,L)} = \frac{1}{2}\frac{\partial}{\partial t}\|P\|^2_{L^2_\rho(0,L)}. \tag{46}$$

The combination of equalities (27)–(46) and the use of Lemma 3 yield

$$\begin{aligned}
&\tfrac{\rho_1}{2}\partial_t^\alpha\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \tfrac{\rho_2}{2}\partial_t^\alpha\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} + \tfrac{\kappa_1}{2}\tfrac{\partial}{\partial t}\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \\
&+\kappa_1\|Y_x\|^2_{L^2_\rho(0,L)} + \kappa_2\|P_x\|^2_{L^2_\rho(0,L)} + \tfrac{1}{2}\tfrac{\partial}{\partial t}\|Y\|^2_{L^2_\rho(0,L)} + \tfrac{1}{2}\tfrac{\partial}{\partial t}\|P\|^2_{L^2_\rho(0,L)} \\
&+\kappa_1\|P\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \\
&+\tfrac{\rho_1}{2}\partial_t^{\alpha+1}\|Y\|^2_{L^2_\rho(0,L)} + \tfrac{\rho_1}{2}\partial_t^{\alpha+1}\|P\|^2_{L^2_\rho(0,L)} \\
\leq\ & \kappa_1(Y_x, \mathcal{I}_x(\xi Y_t))_{L^2_\rho(0,L)} + \kappa_2(P_x, \mathcal{I}_x(\xi P_t))_{L^2_\rho(0,L)} - \kappa_1(P_x, \mathcal{I}_x^2(\xi Y_t))_{L^2_\rho(0,L)} \\
&+\kappa_1(Y_x, \mathcal{I}_x^2(\xi P_t))_{L^2_\rho(0,L)} + \kappa_1(P_x, Y)_{L^2_\rho(0,L)} - \kappa_1(Y_x, P)_{L^2_\rho(0,L)} + (G, P)_{L^2_\rho(0,L)} \\
&-(F, \mathcal{I}_x^2(\xi Y_t))_{L^2_\rho(0,L)} - (G, \mathcal{I}_x^2(\xi P_t))_{L^2_\rho(0,L)} + (F, Y)_{L^2_\rho(0,L)}.
\end{aligned} \tag{47}$$

By applying Cauchy ε inequality $ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad \forall\varepsilon > 0$, and *Poincare*$'$ type inequality $\|\mathcal{J}_x^2(\xi u)\|^2_{L^2(\Omega)} \leqslant \frac{L^2}{2}\|\mathcal{J}_x(\xi u)\|^2_{L^2(\Omega)}$ to the terms of the RHS of (47), we obtain

$$\kappa_1(Y_x, \mathcal{I}_x(\xi Y_t))_{L^2_\rho(0,L)} \leq \frac{\kappa_1\delta_1}{2}\|Y_x\|^2_{L^2_\rho(0,L)} + \frac{\kappa_1 L}{2\delta_1}\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)}, \tag{48}$$

$$\kappa_2(P_x, \mathcal{I}_x(\xi P_t))_{L^2_\rho(0,L)} \leq \frac{\kappa_2\delta_2}{2}\|P_x\|^2_{L^2_\rho(0,L)} + \frac{\kappa_1 L}{2\delta_2}\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)}, \tag{49}$$

$$-\kappa_1(P_x, \mathcal{I}_x^2(\xi Y_t))_{L^2_\rho(0,L)} \leq \frac{\kappa_1\delta_3}{2}\|P_x\|^2_{L^2_\rho(0,L)} + \frac{\kappa_1 L^2}{4\delta_3}\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)}, \tag{50}$$

$$\kappa_1(Y_x, \mathcal{I}_x^2(\xi P_t))_{L^2_\rho(0,L)} \leq \frac{\kappa_1\delta_4}{2}\|Y_x\|^2_{L^2_\rho(0,L)} + \frac{\kappa_1 L^2}{2\delta_4}\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)}, \tag{51}$$

$$\kappa_1(P_x, Y)_{L^2_\rho(0,L)} \leq \frac{\kappa_1\delta_5}{2}\|P_x\|^2_{L^2_\rho(0,L)} + \frac{\kappa_1}{2\delta_5}\|Y\|^2_{L^2_\rho(0,L)}, \tag{52}$$

$$-\kappa_1(Y_x, P)_{L^2_\rho(0,L)} \leq \frac{\kappa_1\delta_6}{2}\|Y_x\|^2_{L^2_\rho(0,L)} + \frac{\kappa_1}{2\delta_6}\|P\|^2_{L^2_\rho(0,L)}, \tag{53}$$

$$-(F, \mathcal{I}_x^2(\xi Y_t))_{L^2_\rho(0,L)} \leq \frac{L}{2\delta_7}\|F\|^2_{L^2(0,L)} + \frac{\delta_7 L^2}{2}\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)}, \tag{54}$$

$$-(G, \mathcal{I}_x^2(\xi P_t))_{L^2_\rho(0,L)} \leq \frac{L}{2\delta_8}\|G\|^2_{L^2(0,L)} + \frac{\delta_8 L^2}{2}\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)}, \tag{55}$$

$$(F,Y)_{L^2_\rho(0,L)} \leq \frac{\delta_9}{2}\|Y\|^2_{L^2_\rho(0,L)} + \frac{1}{2\delta_9}\|F\|^2_{L^2(0,L)}, \tag{56}$$

$$(G,P)_{L^2_\rho(0,L)} \leq \frac{\delta_{10}}{2}\|P\|^2_{L^2_\rho(0,L)} + \frac{1}{2\delta_{10}}\|G\|^2_{L^2(0,L)}. \tag{57}$$

By ignoring the last four terms on the LHS of (47), and inserting (48)–(57) into (47) and choosing $\delta_1 = \delta_6 = \delta_4 = \frac{1}{2}, \delta_3 = \delta_5 = 1/4, \delta_7 = \delta_8 = \delta_9 = \delta_{10} = 1, \delta_2 = \frac{\kappa_1}{\kappa_2}$ we have

$$\begin{aligned} &\partial_t^\alpha \|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \partial_t^\alpha \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} + \tfrac{\partial}{\partial t}\|Y\|^2_{L^2_\rho(0,L)} + \tfrac{\partial}{\partial t}\|P\|^2_{L^2_\rho(0,L)} \\ &+\|Y_x\|^2_{L^2_\rho(0,L)} + \|P_x\|^2_{L^2_\rho(0,L)} + \tfrac{\partial}{\partial t}\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \\ \leq\ & Q^*\Big\{\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} + \|Y\|^2_{L^2_\rho(0,L)} + \|P\|^2_{L^2_\rho(0,L)} \\ &+ \|F\|^2_{L^2(0,L)} + \|G\|^2_{L^2(0,L)}\Big\}, \end{aligned} \tag{58}$$

where

$$Q^* = \frac{\max\left[\frac{3}{4}\kappa_1, 2\kappa_1 + \frac{1}{2}, L\kappa_1 + \frac{3}{2}L^2\kappa_1, \frac{L\kappa_2^2}{2\kappa_1} + L^2\kappa_1 + \frac{L^2}{2}, L, \frac{1+L}{2}\right]}{\min\left[\frac{\rho_1}{2}, \frac{\rho_2}{2}, \frac{\kappa_1}{2}, \frac{1}{2}, \kappa_2\right]}. \tag{59}$$

By using the *Poincare*′ type inequality $\|\mathcal{J}_x(\xi u)\|^2_{L^2(\Omega)} \leqslant \frac{L^2}{2}\|u\|^2_{L^2(\Omega)}$, replacing t by s, and integrating both sides of (58) with respect to s over $[0,t]$, we obtain

$$\begin{aligned} &D^{\alpha-1}\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + D^{\alpha-1}\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \\ &+\|Y_x\|^2_{L^2_\rho(0.,t,L^2_\rho(0,L))} + \|P_x\|^2_{L^2_\rho(0.,t,L^2_\rho(0,L))} + \|Y(x.s)\|^2_{L^2_\rho(0,L)} \\ &+\|P(x.s)\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_x(\xi P_t(.\xi,s))\|^2_{L^2(0,L)} \\ \leq\ & Q^*\Bigg(\int_0^t \Big(\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)}\Big)ds \\ &+ \int_0^t \Big(\|Y\|^2_{L^2_\rho(0,L)} + \|P\|^2_{L^2_\rho(0,L)}\Big)ds + \|F\|^2_{L^2(0.,t,L^2(0,L))} + \|G\|^2_{L^2(0.,t,L^2(0,L))} \\ &+ \tfrac{t^{1-\alpha}L^2}{2(1-\alpha)\Gamma(1-\alpha)}\Big(\|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)}\Big) \\ &+ \|\varphi\|^2_{L^2_\rho(0,L)} + \|f\|^2_{L^2_\rho(0,L)}\Bigg). \end{aligned} \tag{60}$$

We rewrite (60) after discarding the last term on its LHS as

$$\begin{aligned} &D^{\alpha-1}\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + D^{\alpha-1}\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \\ &+\|Y_x\|^2_{L^2_\rho(0.,t,L^2_\rho(0,L))} + \|P_x\|^2_{L^2_\rho(0.,t,L^2_\rho(0,L))} + \|Y(x.s)\|^2_{L^2_\rho(0,L)} \\ &+\|P(x.s)\|^2_{L^2_\rho(0,L)} \\ \leq\ & Q^{**}\Bigg(\int_0^t \Big(\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)}\Big)ds \\ &+ \int_0^t \Big(\|Y\|^2_{L^2_\rho(0,L)} + \|P\|^2_{L^2_\rho(0,L)}\Big)ds + \|F\|^2_{L^2(0.,t,L^2(0,L))} + \|G\|^2_{L^2(0.,t,L^2(0,L))} \\ &+ \|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} + \|\varphi\|^2_{L^2_\rho(0,L)} + \|f\|^2_{L^2_\rho(0,L)}\Bigg), \end{aligned} \tag{61}$$

where

$$Q^{**} = Q^* \max\left\{1, \frac{t^{1-\alpha}L^2}{2(1-\alpha)\Gamma(1-\alpha)}\right\}. \tag{62}$$

The first and second time integrals on the RHS must be removed. In this regard, by keeping only the fifth and sixth terms on the LHS of (61), and applying the Gronwall–Bellman lemma (Lemma 1) with

$$\begin{cases} H(t) = \int_0^t \left(\|Y\|^2_{L^2_\rho(0,L)} + \|P\|^2_{L^2_\rho(0,L)} \right) ds \\ \frac{dH}{dt} = \|Y(x.s)\|^2_{L^2_\rho(0,L)} + \|P(x.s)\|^2_{L^2_\rho(0,L)}, \\ H(0) = 0. \end{cases} \tag{63}$$

we obtain

$$\begin{aligned} H(t) \quad \leq \quad & Q^{**} e^{Q^{**}T} \left(\int_0^t \left(\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \right) ds \right. \\ & + \|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} + \|\varphi\|^2_{L^2_\rho(0,L)} + \|f\|^2_{L^2_\rho(0,L)} \\ & \left. + \|F\|^2_{L^2(0,t;L^2(0,L))} + \|G\|^2_{L^2(0,t;L^2(0,L))} \right). \end{aligned} \tag{64}$$

By dropping the last four terms on the LHS of (61), and using (64), we have

$$\begin{aligned} & D^{\alpha-1} \|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + D^{\alpha-1} \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \\ \leq \quad & Q^{**}(Q^{**} e^{Q^{**}T} + 1) \left(\int_0^t \left(\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \right) ds \right. \\ & + \|F\|^2_{L^2(0,t;L^2(0,L))} + \|G\|^2_{L^2(0,t;L^2(0,L))} + \|\psi\|^2_{L^2(0,L)} \\ & \left. + \|g\|^2_{L^2(0,L)} + \|\varphi\|^2_{L^2_\rho(0,L)} + \|f\|^2_{L^2_\rho(0,L)} \right). \end{aligned} \tag{65}$$

We now can apply Lemma 2 to eliminate the time integral term on the RHS of (65). For this purpose, we let

$$\begin{cases} \mathcal{B}(t) = \int_0^t \left(\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \right) ds \\ \partial_t^\alpha \mathcal{B}(t) = D^{\alpha-1} \|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)} + D^{\alpha-1} \|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)} \\ \mathcal{B}(0) = 0. \end{cases} \tag{66}$$

Then, from (66), it follows that

$$\begin{aligned} \mathcal{B}(t) \quad \leq \quad & \mathcal{V} D^{-1-\alpha} \left\{ \|F\|^2_{L^2(0,t;L^2(0,L))} + \|G\|^2_{L^2(0,t;L^2(0,L))} \right. \\ & \left. + \|\varphi\|^2_{L^2_\rho(0,L)} + \|\psi\|^2_{L^2(0,L)} + \|f\|^2_{L^2_\rho(0,L)} + \|g\|^2_{L^2(0,L)} \right\}, \end{aligned} \tag{67}$$

where

$$\mathcal{V} = \Gamma(\alpha) E_{\alpha,\alpha}(\lambda T^\alpha) \left(\max\left\{ 1, \frac{T^\alpha}{\alpha \Gamma(\alpha)} \right\} \right), \tag{68}$$

with

$$\lambda = Q^{**}(Q^{**} e^{Q^{**}T} + 1). \tag{69}$$

Observe that according to (21), we have

$$\begin{aligned} & D^{-1-\alpha}\Big\{\|F\|^2_{L^2(0,t;L^2(0,L))}+\|G\|^2_{L^2(0,t;L^2(0,L))} \\ & +\|\varphi\|^2_{L^2_\rho(0,L)}+\|\psi\|^2_{L^2(0,L)}+\|f\|^2_{L^2_\rho(0,L)}+\|g\|^2_{L^2(0,L)}\Big\} \\ \leq\ & \frac{T^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}\Big(\|F\|^2_{L^2(0,t;L^2(0,L))}+\|G\|^2_{L^2(0,t;L^2(0,L))} \\ & +\|\varphi\|^2_{L^2_\rho(0,L)}+\|\psi\|^2_{L^2(0,L)}+\|f\|^2_{L^2_\rho(0,L)}+\|g\|^2_{L^2(0,L)}\Big). \end{aligned} \tag{70}$$

We deduce from inequalities (61), (67), and (70), that

$$\begin{aligned} & D^{\alpha-1}\|\mathcal{I}_x(\xi Y_t)\|^2_{L^2(0,L)}+D^{\alpha-1}\|\mathcal{I}_x(\xi P_t)\|^2_{L^2(0,L)}+\|Y_x\|^2_{L^2_\rho(0.,t,L^2_\rho(0,L))} \\ & +\|P_x\|^2_{L^2_\rho(0.,t,L^2_\rho(0,L))}+\|Y(x.s)\|^2_{L^2_\rho(0,L)}+\|P(x.s)\|^2_{L^2_\rho(0,L)} \\ \leq\ & \mathcal{C}^*\Big(\|\psi\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)} \\ & +\|\varphi\|^2_{L^2_\rho(0,L)}+\|f\|^2_{L^2_\rho(0,L)}+\|F\|^2_{L^2(0.,t,L^2(0,L))}+\|G\|^2_{L^2(0.,t,L^2(0,L))}\Big), \end{aligned} \tag{71}$$

where

$$\mathcal{C}^*=\mathcal{V}\max\Big\{1,\frac{T^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}\Big\}. \tag{72}$$

If we drop the first two terms on the LHS of (71) and use the equivalence of the norms

$$\|Y_x\|^2_{L^2_\rho(0,L)}\sim\|Y\|^2_{H^1_\rho(0,L)},\ \text{and}\ \|P_x\|^2_{L^2_\rho(0,L)}\sim\|P\|^2_{H^1_\rho(0,L)}, \tag{73}$$

and we take the supremum with respect to t over $[0,T]$, we get the estimate (23), that is

$$\begin{aligned} & \|Y\|^2_{L^2(0.,T,H^1_\rho(0,L))}+\|P\|^2_{L^2(0.,T,H^1_\rho(0,L))}+\|Y\|^2_{C(0.,T,L^2_\rho(0,L))}+\|P\|^2_{C(0.,T,L^2_\rho(0,L))} \\ \leq\ & \mathcal{C}^*\Big(\|\psi\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)} \\ & +\|\varphi\|^2_{L^2_\rho(0,L)}+\|f\|^2_{L^2_\rho(0,L)}+\|F\|^2_{L^2(0.,T,L^2(0,L))}+\|G\|^2_{L^2(0.,T,L^2(0,L))}\Big). \end{aligned} \tag{74}$$

□

Since $R(\Sigma)\subset E$, we extend Σ so that the estimate (23) is true for the extension, and $R(\overline{\Sigma})=E$.

Proposition 1. *The operator $\Sigma : B\to E$ has a closure $\overline{\Sigma}$ with domain $D(\overline{\Sigma})$.*
The proof can be established as in [24].

Definition 3. *We call the solution of the equation $\overline{\Sigma}\mathcal{F}=H=(H_1,H_2)$ a strong solution of problem (2)–(4).*

The energy inequality (23), can be extended to the following

$$\|\mathcal{F}\|^2_B\leq\mathcal{C}^*\|\overline{\Sigma}\mathcal{F}\|^2_E,\quad\forall\mathcal{F}\in D(\overline{\Sigma}). \tag{75}$$

From (75), we deduce that the operator $\overline{\Sigma}$ is injective and that $\overline{\Sigma}^{-1}:R(\overline{\Sigma})\to B$ is continuous. Thus, the solution of problem (2)–(4), if existing, is unique and depends continuously on the data. We also conclude from (75) that the $R(\overline{\Sigma})\subset E$ is closed and $R(\overline{\Sigma})$ coincides with $\overline{R(\Sigma)}$.

5. Existence of Solution

Theorem 2. *For every* $F, G \in L^2(0,T;L^2(0,L))$, φ, $f \in L^2_\rho(0,L)$, *and* ψ, $g \in L^2(0,L)$, *there exists a unique strong generalized solution* $\mathcal{F} = \left(\overline{\Sigma}\right)^{-1}(H_1, H_2) = \overline{\Sigma^{-1}}(H_1, H_2) \in B$ *of problem (2)–(4), and*

$$\|\mathcal{F}\|^2_B \leq \mathcal{C}^* \|\Sigma\mathcal{F}\|^2_E, \ \forall \mathcal{F} \in D(\overline{\Sigma}).$$

Proof. From what is previously mentioned, it is sufficient to show that $\overline{R(\Sigma)} = E$ for establishing the existence of the generalized strong solution of problem (2)–(4). Let us first prove □

Theorem 3. *Density in special case: If for some function* $\omega(x,t) = (\Gamma_1(x,t), \Gamma_2(x,t)) \in (L^2(0,T;L^2(0,L)))^2$ *and for elements* $\mathcal{F} \in D_0(\Sigma) = \{\mathcal{F} : \mathcal{F} \in D(\Sigma)$ *and* $l_i Y = l_i P = 0$, $i = 1,2\}$, *we have*

$$(\mathcal{F}_1(Y,P), \Gamma_1)_{L^2(0,T;L^2\rho(0,L))} + (\mathcal{F}_2(Y,P), \Gamma_2)_{L^2(0,T;L^2\rho(0,L))} = 0, \tag{76}$$

then $\omega(x,t) = (\Gamma_1(x,t), \Gamma_2(x,t)) = (0,0)$ *a.e in the domain* Q^T.

Proof. The identity (76) can be written as

$$\begin{aligned}
&\int_0^T (\rho_1 \partial_t^{\alpha+1} Y, \Gamma_1)_{L^2\rho(0,L)} dt - \kappa_1 \int_0^T (\tfrac{1}{x}(xY_x)_x, \Gamma_1)_{L^2_\rho(0,L)} dt - \kappa_1 \int_0^T (P_x, \Gamma_1)_{L^2_\rho(0,L)} dt \\
&+ \int_0^T (Y_t, \Gamma_1)_{L^2_\rho(0,L)} dt + \int_0^T (\rho_2 \partial_t^{\alpha+1} P, \Gamma_2)_{L^2_\rho(0,L)} dt - \kappa_2 \int_0^T (\tfrac{1}{x}(xP_x)_x, \Gamma_2)_{L^2_\rho(0,L)} dt \\
&+ \kappa_1 \int_0^T (Y_x, \Gamma_2)_{L^2_\rho(0,L)} dt + \kappa_1 \int_0^T (P, \Gamma_2)_{L^2_\rho(0,L)} dt + \int_0^T (P_t, \Gamma_2)_{L^2_\rho(0,L)} dt \\
= \;\; & 0.
\end{aligned} \tag{77}$$

Suppose that the functions U and V verify the conditions (3) and (4) and the functions $U, V, U_x, V_x, \mathcal{I}_t U, \mathcal{I}_t V, \mathcal{I}_t \mathcal{I}_x^2 V, \mathcal{I}_t \mathcal{I}_x^2 V, \mathcal{I}_t^2 U, \mathcal{I}_t^2 V$, and $\partial_t^{\beta+1} U, \partial_t^{\beta+1} V$ are in $L^2(0,T;L^2(0,L))$, we then let

$$Y(x,t) = \mathcal{I}_t^2 U = \int_0^t \int_0^s U(x,z) dz ds, \ P(x,t) = \mathcal{I}_t^2 V = \int_0^t \int_0^s V(x,z) dz ds, \tag{78}$$

and introduce the functions

$$\Gamma_1(x,t) = \mathcal{I}_t U - \mathcal{I}_x^2(\xi \mathcal{I}_t U), \ \Gamma_2(x,t) = \mathcal{I}_t V - \mathcal{I}_x^2(\xi \mathcal{I}_t V). \tag{79}$$

Equation (77) then reduces to

$$
\begin{aligned}
&\int_0^T(\rho_1\partial_t^{\alpha+1}(\mathcal{I}_t^2U),\mathcal{I}_tU-\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)}dt-\kappa_1\int_0^T(\mathcal{I}_t^2\tfrac{1}{x}(xU_x)_x,\mathcal{I}_tU-\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)}dt\\
&-\kappa_1\int_0^T(\mathcal{I}_t^2V_x,\mathcal{I}_tU-\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)}dt+\int_0^T(\mathcal{I}_tU,\mathcal{I}_tU-\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)}dt\\
&+\int_0^T(\rho_2\partial_t^{\alpha+1}\mathcal{I}_t^2V,\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}dt-\kappa_2\int_0^T(\mathcal{I}_t^2\tfrac{1}{x}(xV_x)_x,\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}dt\\
&+\kappa_1\int_0^T(\mathcal{I}_t^2U_x,\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}dt+\kappa_1\int_0^T(\mathcal{I}_t^2V,\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}dt\\
&+\int_0^T(\mathcal{I}_tV,\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}dt\\
=\;&0.
\end{aligned}
\tag{80}
$$

□

Invoking boundary conditions and calculating different terms by integration by parts, we have

$$
\begin{aligned}
&(\rho_1\partial_t^{\alpha+1}(\mathcal{I}_t^2U),\mathcal{I}_tU-\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)}\\
=\;&(\rho_1\partial_t^{\alpha}\mathcal{I}_tU,\mathcal{I}_tU)_{L^2_\rho(0,L)}+(\rho_1\partial_t^{\alpha}\mathcal{I}_x(\xi\mathcal{I}_tU),\mathcal{I}_x(\xi\mathcal{I}_tU))_{L^2(0,L)},
\end{aligned}
\tag{81}
$$

$$
\begin{aligned}
&-\kappa_1(\mathcal{I}_t^2\tfrac{1}{x}(xU_x)_x,\mathcal{I}_tU-\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)}\\
=\;&\tfrac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2U_x\|^2_{L^2_\rho(0,L)}-\kappa_1(\mathcal{I}_t^2U_x,\mathcal{I}_x(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)},
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
&-\kappa_1(\mathcal{I}_t^2V_x,\mathcal{I}_tU-\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)}\\
=\;&-\kappa_1(\mathcal{I}_t^2V_x,\mathcal{I}_tU)_{L^2_\rho(0,L)}+\kappa_1(\mathcal{I}_t^2V_x,\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)},
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
&(\mathcal{I}_tU,\mathcal{I}_tU-\mathcal{I}_x^2(\xi\mathcal{I}_tU))_{L^2_\rho(0,L)}\\
=\;&\|\Im_tU\|^2_{L^2_\rho(0,L)}+\|\mathcal{I}_x(\xi\mathcal{I}_tU)\|^2_{L^2(0,L)},
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&(\rho_2\partial_t^{\alpha+1}(\mathcal{I}_t^2V),\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}\\
=\;&(\rho_2\partial_t^{\alpha}\mathcal{I}_tV,\mathcal{I}_tV)_{L^2_\rho(0,L)}+(\rho_2\partial_t^{\alpha}\mathcal{I}_x(\xi\mathcal{I}_tV),\mathcal{I}_x(\xi\mathcal{I}_tV))_{L^2(0,L)},
\end{aligned}
\tag{85}
$$

$$
\begin{aligned}
&-\kappa_2(\mathcal{I}_t^2\tfrac{1}{x}(xV_x)_x,\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}\\
=\;&\tfrac{\kappa_2\partial}{2\partial t}\|\mathcal{I}_t^2V_x\|^2_{L^2_\rho(0,L)}-\kappa_2(\mathcal{I}_t^2V_x,\mathcal{I}_x(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)},
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
&\kappa_1(\mathcal{I}_t^2U_x,\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}\\
=\;&\kappa_1(\mathcal{I}_t^2U_x,\mathcal{I}_tV)_{L^2_\rho(0,L)}-\kappa_1(\mathcal{I}_t^2U_x,\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)},
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
&\kappa_1(\mathcal{I}_t^2V,\mathcal{I}_tV-\mathcal{I}_x^2(\xi\mathcal{I}_tV))_{L^2_\rho(0,L)}\\
=\;&\tfrac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2V\|^2_{L^2_\rho(0,L)}+\tfrac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2\mathcal{I}_x(\xi V)\|^2_{L^2(0,L)}
\end{aligned}
\tag{88}
$$

$$\begin{aligned} & (\mathcal{I}_t V, \mathcal{I}_t V - \mathcal{I}_x^2(\xi \mathcal{I}_t V))_{L^2_\rho(0,L)} \\ = \; & \|\mathcal{I}_t V\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_t \mathcal{I}_x(\xi V)\|^2_{L^2(0,L)}. \end{aligned} \quad (89)$$

The combination of (80)–(89) and application of Lemma 2 to (81) and (85) yield

$$\begin{aligned} & \tfrac{\rho_1}{2}\partial_t^\alpha \|\mathcal{I}_t U\|^2_{L^2_\rho(0,L)} + \tfrac{\rho_1}{2}\partial_t^\alpha \|\mathcal{I}_x(\xi \mathcal{I}_t U)\|^2_{L^2(0,L)} \\ & + \tfrac{\rho_2}{2}\partial_t^\alpha \|\mathcal{I}_t V\|^2_{L^2_\rho(0,L)} + \tfrac{\rho_2}{2}\partial_t^\alpha \|\mathcal{I}_x(\xi \mathcal{I}_t V)\|^2_{L^2(0,L)} \\ & + \tfrac{\kappa_1 \partial}{2\partial t} \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \tfrac{\kappa_2 \partial}{2\partial t} \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} \\ & + \|\Im_t U\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_x(\xi \mathcal{I}_t U)\|^2_{L^2(0,L)} \\ & + \tfrac{\kappa_1 \partial}{2\partial t} \|\mathcal{I}_t^2 V\|^2_{L^2_\rho(0,L)} + \tfrac{\kappa_1 \partial}{2\partial t} \|\mathcal{I}_t^2 \mathcal{I}_x(\xi V)\|^2_{L^2(0,L)} \\ & \|\mathcal{I}_t V\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_t \mathcal{I}_x(\xi V)\|^2_{L^2(0,L)} \\ \le \; & \kappa_1(\mathcal{I}_t^2 U_x, \mathcal{I}_x(\xi \mathcal{I}_t U))_{L^2_\rho(0,L)} + \kappa_1(\mathcal{I}_t^2 V_x, \mathcal{I}_t U)_{L^2_\rho(0,L)} \\ & - \kappa_1(\mathcal{I}_t^2 V_x, \mathcal{I}_x^2(\xi \mathcal{I}_t U))_{L^2_\rho(0,L)} + \kappa_2(\mathcal{I}_t^2 V_x, \mathcal{I}_x(\xi \mathcal{I}_t V))_{L^2_\rho(0,L)} \\ & - \kappa_1(\mathcal{I}_t^2 U_x, \mathcal{I}_t V)_{L^2_\rho(0,L)} + \kappa_1(\mathcal{I}_t^2 U_x, \mathcal{I}_x^2(\xi \mathcal{I}_t V))_{L^2_\rho(0,L)} \end{aligned} \quad (90)$$

We now estimate all terms on the RHS of (90) with the help of Cauchy epsilon inequality

$$\kappa_1(\mathcal{I}_t^2 U_x, \mathcal{I}_x(\xi \mathcal{I}_t U))_{L^2_\rho(0,L)} \le \frac{\varepsilon_1 \kappa_1}{2} \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \frac{L\kappa_1}{2\varepsilon_1} \|\mathcal{I}_x(\xi \mathcal{I}_t U)\|^2_{L^2(0,L)}, \quad (91)$$

$$\kappa_1(\mathcal{I}_t^2 V_x, \mathcal{I}_t U)_{L^2_\rho(0,L)} \le \frac{\kappa_1}{2\varepsilon_2} \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} + \frac{\varepsilon_2 \kappa_1}{2} \|\mathcal{I}_t U\|^2_{L^2_\rho(0,L)}, \quad (92)$$

$$-\kappa_1(\mathcal{I}_t^2 V_x, \mathcal{I}_x^2(\xi \mathcal{I}_t U))_{L^2_\rho(0,L)} \le \frac{\varepsilon_3 \kappa_1}{2} \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} + \frac{L^2 \kappa_1}{4\varepsilon_3} \|\mathcal{I}_x(\xi \mathcal{I}_t U)\|^2_{L^2(0,L)} \quad (93)$$

$$\kappa_2(\mathcal{I}_t^2 V_x, \mathcal{I}_x(\xi \mathcal{I}_t V))_{L^2_\rho(0,L)} \le \frac{\varepsilon_4 \kappa_2}{2} \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} + \frac{L\kappa_1}{2\varepsilon_4} \|\mathcal{I}_x(\xi \mathcal{I}_t V)\|^2_{L^2(0,L)}, \quad (94)$$

$$-\kappa_1(\mathcal{I}_t^2 U_x, \mathcal{I}_t V)_{L^2_\rho(0,L)} \le \frac{\kappa_1}{2\varepsilon_5} \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \frac{\varepsilon_5 \kappa_1}{2} \|\mathcal{I}_t V\|^2_{L^2_\rho(0,L)}, \quad (95)$$

$$\kappa_1(\mathcal{I}_t^2 U_x, \mathcal{I}_x^2(\xi \mathcal{I}_t V))_{L^2_\rho(0,L)} \le \frac{\varepsilon_6 \kappa_1}{2} \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \frac{L\kappa_1}{2\varepsilon_6} \|\mathcal{I}_x(\xi \mathcal{I}_t V)\|^2_{L^2(0,L)}. \quad (96)$$

By choosing $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 1$, and $\varepsilon_2 = \varepsilon_5 = \frac{2}{\kappa_1}$ and taking into account (91)–(96), the inequality (90) reduces to

$$\begin{aligned} & \tfrac{\partial}{\partial t} \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \tfrac{\partial}{\partial t} \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} + \partial_t^\alpha \|\mathcal{I}_x(\xi \mathcal{I}_t U)\|^2_{L^2(0,L)} \\ & + \partial_t^\alpha \|\mathcal{I}_x(\xi \mathcal{I}_t V)\|^2_{L^2(0,L)} + \tfrac{\partial}{\partial t} \|\mathcal{I}_t^2 V\|^2_{L^2_\rho(0,L)} \\ & + \tfrac{\partial}{\partial t} \|\mathcal{I}_t^2 \mathcal{I}_x(\xi V)\|^2_{L^2(0,L)} + \partial_t^\alpha \|\mathcal{I}_t U\|^2_{L^2_\rho(0,L)} + \partial_t^\alpha \|\mathcal{I}_t V)\|^2_{L^2_\rho(0,L)} \\ \le \; & E^* \Big(\|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_x(\xi \mathcal{I}_t U)\|^2_{L^2(0,L)} \\ & + \|\mathcal{I}_x(\xi \mathcal{I}_t V)\|^2_{L^2(0,L)} \Big), \end{aligned} \quad (97)$$

where

$$E^* = \frac{\max\left\{ 1 + \frac{\kappa_1}{2} + \frac{L\kappa_1\kappa_2}{8}, \frac{1}{L} + \frac{\kappa_1^2}{4} + \frac{\kappa_1}{2}, \frac{L^2\kappa_1}{4} + \frac{L\kappa_1}{2} \right\}}{\min\left\{ \frac{\rho_1}{2}, \frac{\rho_2}{2}, \frac{\kappa_1}{2}, \frac{\kappa_2}{2} \right\}}. \quad (98)$$

Now, by discarding the last four terms on the LHS of the inequality (97), then integrating with respect to time, we have

$$\begin{aligned} & \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} + D_t^{\alpha-1}\|\mathcal{I}_x(\xi\mathcal{I}_t U)\|^2_{L^2(0,L)} \\ & D_t^{\alpha-1}\|\mathcal{I}_x(\xi\mathcal{I}_t V)\|^2_{L^2(0,L)} \\ \leq\ & E^*\left(\int_0^t \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)}d\tau + \int_0^t \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)}d\tau + \int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t U)\|^2_{L^2(0,L)}d\tau \right. \\ & \left. + \int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t V)\|^2_{L^2(0,L)}d\tau\right). \end{aligned} \tag{99}$$

If we omit the last two terms on the LHS of (99), and use the Gronwall–Bellman lemma by taking

$$\begin{cases} \mathcal{X}(t) = \int_0^t \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)}d\tau + \int_0^t \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)}d\tau \\ \frac{\partial\mathcal{X}(t)}{\partial t} = \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} \\ \mathcal{X}(0) = 0, \end{cases} \tag{100}$$

we obtain

$$\mathcal{X}(t) \leq E^* e^{TE^*}\left(\int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t U)\|^2_{L^2(0,L)}d\tau + \int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t V)\|^2_{L^2(0,L)}d\tau\right) \tag{101}$$

Consequently, inequality (99) reduces to

$$\begin{aligned} & \|\mathcal{I}_t^2 U_x\|^2_{L^2_\rho(0,L)} + \|\mathcal{I}_t^2 V_x\|^2_{L^2_\rho(0,L)} + D_t^{\alpha-1}\|\mathcal{I}_x(\xi\mathcal{I}_t U)\|^2_{L^2(0,L)} \\ & D_t^{\alpha-1}\|\mathcal{I}_x(\xi\mathcal{I}_t V)\|^2_{L^2(0,L)} \\ \leq\ & E^{**}\left(\int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t U)\|^2_{L^2(0,L)}d\tau + \int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t V)\|^2_{L^2(0,L)}d\tau\right), \end{aligned} \tag{102}$$

where $E^{**} = E^*(1 + E^* e^{TE^*})$.

Next, if we drop the first two terms on the LHS of (99) and let

$$\begin{aligned} d(t) &= \int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t U)\|^2_{L^2(0,L)}d\tau + \int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t V)\|^2_{L^2(0,L)}d\tau \\ \partial_t^\alpha d(t) &= D_t^{\alpha-1}\|\mathcal{I}_x(\xi\mathcal{I}_t U)\|^2_{L^2(0,L)} + D_t^{\alpha-1}\|\mathcal{I}_x(\xi\mathcal{I}_t V)\|^2_{L^2(0,L)}, \\ d(0) &= 0, \end{aligned} \tag{103}$$

then application of Lemma 2 asserts that

$$\begin{aligned} d(t) &= \int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t U)\|^2_{L^2(0,L)}d\tau + \int_0^t \|\mathcal{I}_x(\xi\mathcal{I}_t V)\|^2_{L^2(0,L)}d\tau \\ &\leq d(0)E_\alpha(E^{**}t^\alpha) + \Gamma(\alpha)E_{\alpha,\alpha}(E^{**}t^\alpha)D^{\alpha-1}(0) = 0. \end{aligned} \tag{104}$$

It then follows from (99), (102), and (104) that $U = 0$, $V = 0$. Consequently, $\omega(x,t) = (\Gamma_1(x,t), \Gamma_2(x,t)) = (0,0)$ a.e in Q^T.

General case for density: Since E is a Hilbert space, $\overline{R(\Sigma)} = E$ is equivalent to $R(\Sigma)^\perp = \{0\}$. That is, we have to show that the orthogonal complement of the range of the operator Σ reduces to zero, which is again equivalent to $(\Sigma\mathcal{F}, \Psi)_E = 0$, for all $\mathcal{F} \in B$, and $\Psi \in E$,then $\Psi = (\Psi_1, \Psi_2) = \{(\omega_1, \omega_2, \omega_3), (\omega_4, \omega_5, \omega_6)\} = (0, 0)$. Thus, suppose that for some element $\Psi = (\Psi_1, \Psi_2) = \{(\omega_1, \omega_2, \omega_3), (\omega_4, \omega_5, \omega_6)\} \in R(\Sigma)^{\perp}$, the inner product

$$\begin{aligned} & (\Sigma\mathcal{F}, \Psi)_E \\ = & (\{\mathcal{F}_1(Y,P), \mathcal{F}_2(Y,P\}, \{\Psi_1, \Psi_2\})_E \\ = & (\{\mathcal{F}_1(Y,P), l_1 Y, l_2 Y\}, \{\mathcal{F}_2(Y,P), l_1 P, l_2 P\}\}, \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}\})_E \\ = & (\mathcal{F}_1(Y,P), \omega_1)_{L^2(Q^T)} + (l_1 Y, \omega_2)_{L^2_\rho(0,L)} + (l_2 Y, \omega_3)_{L^2(0,L)} \\ & +(\mathcal{F}_2(Y,P), \omega_4)_{L^2(Q^T)} + (l_1 P, \omega_5)_{L^2_\rho(0,L)} + (l_2 P, \omega_6)_{L^2(0,L)} \\ = & 0, \end{aligned} \tag{105}$$

where $\mathcal{F} \in B$, we have to show that $\Psi = 0$.

Let $\mathcal{F} \in D_0(\Sigma)$, then Equation (105) becomes

$$(\mathcal{F}_1(Y,P), \omega_1)_{L^2(Q^T)} + (\mathcal{F}_2(Y,P), \omega_4)_{L^2(Q^T)} = 0. \tag{106}$$

From Theorem 3, it follows from (106) that $\omega_1 = \omega_4 = 0$. Then, Equation (105) takes the form

$$(l_1 Y, \omega_2)_{L^2_\rho(0,L)} + (l_2 Y, \omega_3)_{L^2(0,L)} + (l_1 P, \omega_5)_{L^2_\rho(0,L)} + (l_2 P, \omega_6)_{L^2(0,L)} = 0. \tag{107}$$

Since all terms in (107) vanish independently, and on the other hand the sets $R(l_1), R(l_2)$ are, respectively, everywhere dense in $L^2_\rho(0, L)$, and $L^2(0, L)$, then it follows from (107) that $\omega_2 = \omega_3 = \omega_5 = \omega_6 = 0$. Consequently $\Psi = 0$; that is, $R(\Sigma)^{\perp} = \{0\}$. Thus, $\overline{R(\Sigma)} = E$. This achieved the proof of Theorem 2.

6. Conclusions

The present article covers the investigation of a non-homogeneous fractional order Timoshenko system with both equations having frictional damping terms. The system of equations is supplemented by initial conditions, two Neumann conditions, and two nonlocal boundary conditions of integral type. The well-posedness of problem (2)–(4) was investigated. The posed problem can be considered as a generalization of the Timoshenko system, where the second order time derivatives are replaced by the Caputo fractional order time derivatives, and the special second order differential operators in both equations are replaced by the Bessel operator. In our proofs, we mainly rely on the operator theory approach. Our contribution develops the energy methods for the fractional case of several classes of initial boundary value problems, particularly for Timoshenko systems.

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