Lecture notes on: Nonlinear optimality in one dimension

The main aim of this chapter is to:

- Calculate the roots of function and its derivative.
- Calculate the critical points of a function on where maximum or minimum value of the function occurred.
- Applying some mathematical procedures such as direct solution, Bisection and Newton Raphson to solve nonlinear programming problems in one variable.

Nonlinear function: It is a type of function that cannot be written in the following form:

$$f(\bar{x}) = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \sum_{i=1}^{n} c_ix_i,$$

$$\bar{x} = (x_1, x_2, \ldots, x_n)$$

Where, $c_i, i = 1,2,\ldots,n$ is constant.

Examples of nonlinear functions:

(i) $f(x) = x\sin(x)$,

(ii) $f(x) = x^2 + 2x - 1$,

(iii) $f(x) = \frac{1}{x} + \log(x^2 + 1)$,

(iv) $f(x) = x\sin(x) + \cos(x) + e^x$,

(v) $f(x) = \ln(\sqrt{x}) + 3x^3 - \tan(x)$

Nonlinear programming: Suppose we have the following program:

$$\begin{aligned} &\text{Max (or Min)} && f(x) \\ &\text{s.t.} && g_i(x) \le b_i;\ i = 1,2,\ldots,m \\ & && h_j(x) = a_j;\ j = 1,2,\ldots,l \\ & && a_j, b_i \ge 0. \end{aligned}$$

This program is called nonlinear if one of the following conditions is satisfied:

- The objective function f(x) is nonlinear.
- The objective function and one of the constraints are nonlinear.
- The objective function and constraints are nonlinear.

Types of Nonlinear programming

- Constrained programming: it has at least one constraint.
- Unconstrained programming: it has no constraints.

Nonlinear optimality in one variable

We study in this chapter the following type of unconstrained optimization problem

$$\begin{aligned} &\text{Max (or Min)} \quad f(x) \\ &\text{s.t.} \qquad\qquad a \le x \le b \end{aligned}$$

We assume that the function f(x) is well-defined on the closed interval [a,b]. We assume also that it is continuous on its domain. Its derivatives are existed on the open interval (a,b).

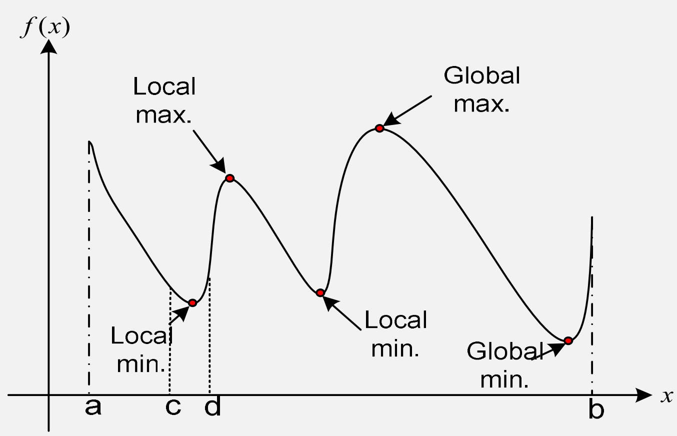
local and global minimum/maximum points

Local minimum point: A point $x^* \in [a,b]$ is said to be a local minimum point for the function $f: [a,b] \to \mathbb{R}$ if $f(x^*) \le f(x) \;\; \forall \;\; x \in (c,d) \subseteq [a,b]$.

Local maximum point: A point $x^* \in [a,b]$ is said to be a local maximum point for the function $f: [a,b] \to \mathbb{R}$ if $f(x^*) \ge f(x) \;\; \forall \;\; x \in (c,d) \subseteq [a,b]$

Global minimum point: A point $x^* \in [a,b]$ is said to be a global minimum point for the function $f: [a,b] \to \mathbb{R}$ if $f(x^*) \le f(x) \;\; \forall \;\; x \in [a,b]$.

Global maximum point: A point $x^* \in [a,b]$ is said to be a local maximum point for the function $f: [a,b] \to \mathbb{R}$ if $f(x^*) \ge f(x) \;\; \forall \;\; x \in [a,b]$



Unimodal function: A function $f: [a,b] \to \mathbb{R}$ is said to be unimodal function if one of the following conditions is satisfied:

(i) $\exists\, x^* \in [a,b], \forall\, x_1, x_2 \in [a,b]$ if $x^* < x_1 < x_2 \Rightarrow f(x^*) < f(x_1) < f(x_2)$ or

(ii) $\exists\, x^* \in [a,b], \forall\, x_3, x_4 \in [a,b]$ if $x^* > x_4 > x_3 \Rightarrow f(x^*) < f(x_4) < f(x_3)$

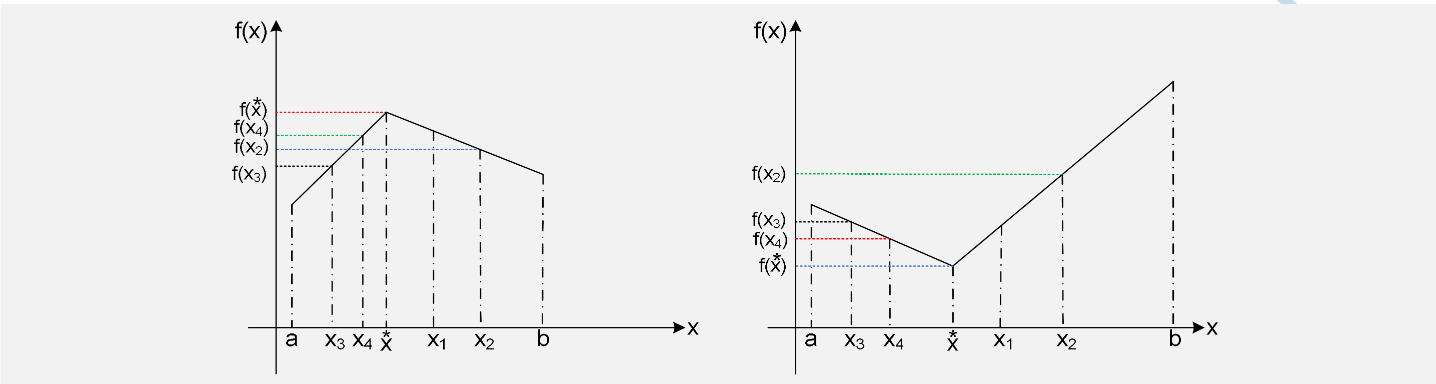
and in this case the point x^* is global minimum point.

Or one of the following conditions is satisfied:

(i) $\exists\, x^* \in [a,b], \forall\, x_1, x_2 \in [a,b]$ if $x^* < x_1 < x_2 \Rightarrow f(x^*) > f(x_1) > f(x_2)$ or

(ii) $\exists\, x^* \in [a,b], \forall\, x_3, x_4 \in [a,b]$ if $x^* > x_4 > x_3 \Rightarrow f(x^*) > f(x_4) > f(x_3)$

and in this case the point x^* is global maximum point.



Note: Unimodal function may be discontinuous, undifferentiable.

Important definitions

(1) Root point: for a function f(x), $f: \mathbb{R} \to \mathbb{R}$ the value of the x that makes f(x)=0 is called a root point.

(2) fixed point: for a function f(x), $f: \mathbb{R} \to \mathbb{R}$ the value of the x that makes f(x)=x is called a fixed point.

(3) Stationary point: for a function f(x), $f: \mathbb{R} \to \mathbb{R}$ the value of the x that makes $f'(x) = 0$ is called a Stationary point.

(4) local minimum point: for a function f(x), $f: [a,b] \to \mathbb{R}$ the point $x_o \in (a,b)$ is a local minimum point if $f'(x_o) = 0$ and $f''(x_o) > 0$.

(4) local maximum point: for a function f(x), $f: [a,b] \to \mathbb{R}$ the point $x_o \in (a,b)$ is a local maximum point if $f'(x_o) = 0$ and $f''(x_o) < 0$.

(5) Saddle point: for a function f(x), $f: [a,b] \to \mathbb{R}$ the point $x_o \in (a,b)$ is a saddle point if $f'(x_o) = 0$ and it is neither minimum nor maximum.

(6) Inflection point: for a function f(x), $f: [a,b] \to \mathbb{R}$ the point $x_o \in (a,b)$ is an inflection point if $f'(x_o) = 0$ and $f''(x_o - \epsilon) f''(x_o + \epsilon) < 0$ where $\epsilon > 0$.

Example

Let $f(x) = x^2 - 2x$

$x = 3$ is fixed point

$f(3) = 3^2 - 6 = 3$

$x = 2$ is root point

$x^2 - 2x = 0 \Longrightarrow x(x-2) = 0 \Longrightarrow x = 0 \text{ or } x = 2$

$x = 1$ is stationary point

$f'(x) = 2x - 2 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

Theory: For the functions with one variable,
(1) If x_o is saddle point, then it is an inflection point.
(2) If x_o is an inflection point, then it is not necessary to be a saddle point.

Optimality conditions

For a function f(x), $f: [a, b] \to \mathbb{R}$, $x_o \in [a, b], f'(x_o) = f''(x_o) = 0$ and $f^n(x_o) \neq 0$ where n is the high non-zero ranked derivative then:

(1) x_o is saddle point if n is odd.
(2) x_o is minimum point if n is even and $f^n(x_o) > 0$.
(3) x_o is maximum point if n is even and $f^n(x_o) < 0$.

Ex: Discuss the properties of the following function: $f(x) = x^5 - 2x^3$.

Let $f(x) = 0 \Rightarrow x^5 - 2x^3 = x^3(x^2 - 2) = 0 \Longrightarrow$ So the roots are $x = 0, \pm\sqrt{2}$

Let $f(x) = x \Rightarrow x^5 - 2x^3 - x = x(x^4 - 2x^2 - 1) = 0 \Longrightarrow x = 0$ or $x^4 - 2x^2 - 1 = 0$

$x^4 - 2x^2 - 1 = 0 \Rightarrow x^2 = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2} \Rightarrow x^2 = 1 + \sqrt{2}$ since $1 - \sqrt{2} < 0 \Rightarrow x = \pm\sqrt{1 + \sqrt{2}}$

So, the fixed points are: $x = 0, \pm\sqrt{1 + \sqrt{2}}$

Let $f'(x) = 0 \Rightarrow 5x^4 - 6x^2 = x^2(5x^2 - 6) = 0 \Longrightarrow x = 0$ or $5x^2 = 6 \Rightarrow x = \pm\sqrt{6/5}$

The stationary points become $x = 0, \pm\sqrt{6/5}$

$f''(x) = 20x^3 - 12x \Rightarrow f''(0) = 0 \Rightarrow f'''(x) = 60x^2 - 12 \Rightarrow f'''(0) = -12 \neq 0,$

$f''\left(\sqrt{6/5}\right) \cong 13.15 > 0,$

$f''\left(-\sqrt{6/5}\right) \cong -13.15 < 0$

The point $x = 0$ is saddle point and hence it is inflection point. The point $x = \sqrt{6/5}$ is a minimum point while $x = -\sqrt{6/5}$ is a maximum point.

Ex: Discuss the properties of the following function: $f(x) = ax + \frac{b}{x}$ where x>0, a and b are constants.

Let $f(x) = 0 \Rightarrow ax + \frac{b}{x} = \frac{ax^2+b}{x} = 0 \Longrightarrow ax^2 + b = 0$. This equation has no real solutions. So, there are no real roots.

Let $f(x) = x \Rightarrow ax + \dfrac{b}{x} = x \Rightarrow (1-a)x^2 - b = 0 \Longrightarrow \begin{cases} \text{if } \ 0 < a < 1 \Rightarrow x = \sqrt{\dfrac{b}{1-a}} \\ \text{if } \ a > 1 \quad\quad \Rightarrow x \notin \mathbb{R} \end{cases}$

So, the function has no real fixed points at $a > 1$. It has a unique fixed value at $0 < a < 1$

Let $f'(x) = 0 \Rightarrow a - \frac{b}{x^2} = 0 \Longrightarrow x^2 = \frac{b}{a} \Rightarrow x = \pm\sqrt{b/a}$ which represents the unique stationary point.

$$f''(x) = \frac{2b}{x^3} \quad \Rightarrow f''\left(\sqrt{b/a}\right) = 2a\sqrt{\frac{a}{b}} > 0$$

$$f_{min} = a\left(\sqrt{b/a}\right) + \frac{b}{\sqrt{b/a}} = 2\sqrt{ab}$$

So, $x = \sqrt{b/a}$ is a minimum point. while $x = -\sqrt{b/a}$ is a maximum point.

Methods of solving nonlinear programming problems in one variable

(1) Direct solution method.
(2) Elimination methods: we focus on the Bisection method in this type.
(3) Interpolation methos: we highlight only Newton-Raphson one.

[1] Direct solution method: Suppose the following nonlinear programming problem:

$$\begin{aligned} &\text{Max (or Min)} \quad f(x) \\ &\text{s.t.} \quad\quad\quad\quad a \le x \le b \end{aligned}$$

Where, f: $[a, b] \rightarrow \mathbb{R}$. The method's steps are given as follows.

Step1: Calculate all the stationary points for the function f(x), i.e., $f'(x) = 0$.
Step2: evaluate all the value of f(x) at the stationary points including f(a) and f(b).
Step3: The optimum solution is the solution giving minimum value or maximum value for the function (where the function is to be maximized or minimized).

Drawbacks of this method

- The method fails to attain the optimum solution if the objective function is not continuous on its domain.
- It also fails to get the optimum solution if the objective function is not differentiable.
- If the stationary points are impossible to obtain, we cannot apply the method.

Ex: Find the optimum solution for the following nonlinear optimization problem:

$$\begin{aligned} &\text{Max} \quad f(x) = x(5\pi - x) \\ &\text{s.t} \quad\ \ 0 \le x \le 20\,;\ \pi \cong 3.14 \end{aligned}$$

Let $f'(x) = 0 \Rightarrow 5\pi - 2x = 0 \Longrightarrow x = \dfrac{5\pi}{2} \cong 7.85 \in [0,20]$

x	0	7.85	20
f(x)	0	61.69	−85.84

$\Rightarrow f_{max} = 61.69$ at $x \cong 7.85$

Ex: Find the optimum solution for the following nonlinear optimization problem:

$$\begin{aligned} &\text{Max} \quad f(x) = -x^3 + 3x^2 + 9x + 10 \\ &\text{s.t} \quad\ -2 \le x \le 4 \end{aligned}$$

Let $f'(x) = 0 \Rightarrow -3x^2 + 6x + 9 = 0 \Longrightarrow (x+1)(x-3) = 0 \ \Rightarrow x = -1,3 \in [-2,4]$

x	−2	−1	3	4
f(x)	12	5	37	30

$\Rightarrow f_{max} = 37$ at $x \cong 3$

Ex: Find the optimum solution for the following nonlinear optimization problem:

$$\begin{aligned} &\text{Max} \quad f(x) = x^4 - 16x^3 + 91x^2 - 216x + 180 \\ &\text{s.t} \quad\ 3.2 \le x \le 5 \end{aligned}$$

Let $f'(x) = 0 \Rightarrow f'(x) = 4x^3 - 48x^2 + 182x - 216 = 0$

To get the roots for the above polynomial we test the factors of 216 as follows.

$216 = 2 \times 3 \times 4 \times 9$ but $2,3,9 \notin [3.2,5]$ so, we have only x=4.

x	3.2	4	5
f(x)	1.2096	4	0

$\Rightarrow f_{max} = 4$ at $x = 4$

Ex: Find the optimum solution for the following nonlinear optimization problem:

$$\text{Max} \quad f(x) = (\ln(x))^3 - 2(\ln(x))^2 + \ln(x)\,,$$
$$\text{s.t} \quad 1 \le x \le e^3; \;\; e \cong 2.72$$

Let $y = \ln(x) \Rightarrow$ if $1 \le x \le e^3 \Rightarrow 0 \le y \le 3, g(y) = y^3 - 2y^2 + y,$

$\Rightarrow g'(y) = 3y^2 - 4y + 1, g''(y) = 6y - 4,$

Let $g'(y) = 0 \Rightarrow 3y^2 - 4y + 1 = (3y-1)(y-1) = 0 \;\; \Rightarrow y = \frac{1}{3}, 1 \in [0,3]$

$g''\left(y = \frac{1}{3}\right) = 2 - 4 < 0, g''(y = 1) = 6 - 4 > 0$

So, at $y = \frac{1}{3}$ $g(y)$ has maximum value.

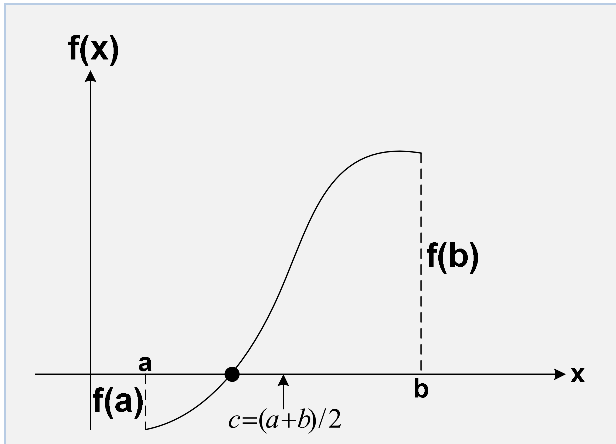
y	0	$\frac{1}{3}$	1	3
g(y)	0	$\frac{4}{27}$	0	11

$\Rightarrow g_{max} = 11$ at $y = 3$, at $y = 3 \Rightarrow 3 = \ln(x) \Rightarrow x = e^3$ and $f_{max}(e^3) = 12$

Bisection method

Finding the root of a function

Let us suppose that the function $f: [a, b] \to \mathbb{R}$ is defined and continuous on its domain [a,b]. It is also differentiable on the open interval (a,b). If f(a)f(b)<0 then the function may have a root.

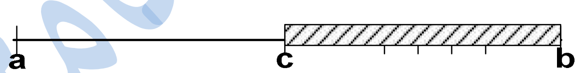


Bisection steps

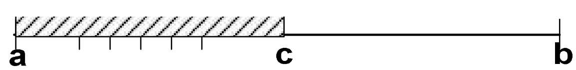
Step 1: calculate the middle of the interval [a,b], say, $c = \frac{a+b}{2}$. If $|f(c)| \leq \epsilon, \epsilon > 0$ (very small value) this means that we reach the root, $x^* = c$. But if $|f(c)| \nleq \epsilon$ we go to step 2.

Step2:

Case 1: if $f(a)f(c) < 0$ this means that the approximate root will be in the interval [a,c] and we reject the interval [b,c]. Then repeat step 1 until we reach the approximate root.

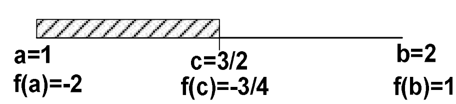


Case 2: if $f(b)f(c) < 0$ this means that the approximate root will be in the interval [c,b] and we reject the interval [a,c]. Then repeat step 1 until we reach the approximate root.



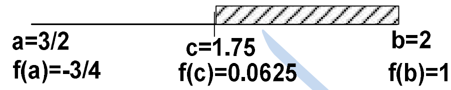
Ex: Find an approximate value for the root of $f(x) = x^2 - 3$ in [1,2] with $\epsilon = 10^{-2}$

a	b	c	f(a)	f(b)	f(c)	|f(c)|
1	2	3/2	−2	1	−3/4	**3/4**

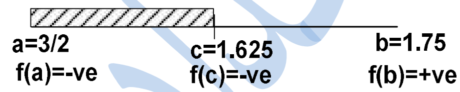


It is clear that $|f(c)| \nleq \epsilon$ and hence $c = \frac{3}{2}$ is not a root. One can see that $f(b)f(c) < 0$ and then we exclude the interval $[a, c]$.

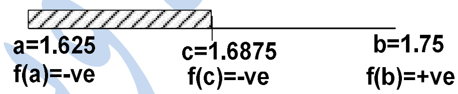
a	b	c	f(a)	f(b)	f(c)	|f(c)|
3/2	2	1.75	−3/4	1	0.0625	**0.0625**



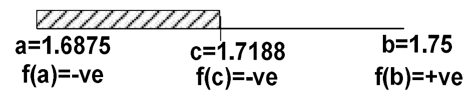
a	b	c	f(a)	f(b)	f(c)	|f(c)|
1.5	1.75	1.625	0.0625	1	−0.3594	**0.3594**



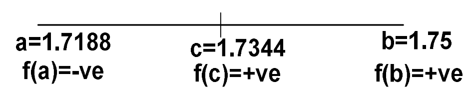
a	b	c	f(a)	f(b)	f(c)	|f(c)|
1.625	1.75	1.6875	−0.3594	0.0625	−0.1523	**0.1523**



a	b	c	f(a)	f(b)	f(c)	|f(c)|
1.6875	1.75	1.7188	−0.1523	0.0625	−0.0459	**0.0459**



a	b	c	f(a)	f(b)	f(c)	|f(c)|
1.7188	1.75	1.7344	−0.0459	0.0625	0.00814	**0.00814**



Now, $|f(c)| < \epsilon$ and then $c = 1.7344$ is an approximate root. The above solution can be summarized in the following table.

n	a	b	c	f(a)	f(b)	f(c)	|f(c)|
1	1	2	1.5	−2	1	−0.75	$\nleq \epsilon$
2	1.5	2	1.75	−0.75	1	0.0625	$\nleq \epsilon$
3	1.5	1.75	1.625	0.0625	1	−0.3594	$\nleq \epsilon$
4	1.625	1.75	1.6875	−0.3594	0.0625	−0.1523	$\nleq \epsilon$
5	1.6875	1.75	1.7188	−0.1523	0.0625	−0.0459	$\nleq \epsilon$
6	1.7188	1.75	1.7344	−0.0459	0.0625	0.00814	$< \epsilon$

Important notes

- This approach is very slow in reaching the approximate root. The number of iterations is very large.
- The root in some problems can be missed.

Ex: Find the third approximate root for $f(x) = \sqrt{x} - \cos(x)$ in [0,1].

n	a	b	c	f(a)	f(b)	f(c)
1	0	1	0.5	-1	$\sqrt{1} - \cos\left(1 \times \frac{180}{\pi}\right) = 0.4597$	$\sqrt{\mathbf{0.5}} - \mathbf{cos}\left(\mathbf{0.5} \times \frac{\mathbf{180}}{\boldsymbol{\pi}}\right) = \mathbf{-0.17048}$
2	0.5	1	0.75	$\mathbf{-0.17048}$	0.4597	$\mathbf{0.13434}$
3	0.5	0.75	0.625	$\mathbf{-0.17048}$	$\mathbf{0.13434}$	$\mathbf{-0.02039}$

Then $p_3 = 0.625$.

Ex: Find the fourth approximate root for $f(x) = 3(x+1)(x-0.5)(x-1)$ in $[-2,1.5]$.

n	a	b	c	f(a)	f(b)	f(c)
1	-2	1.5	-0.25	-22.5	3.75	$\mathbf{2.1094}$
2	-2	-0.25	-1.125	-22.5	$\mathbf{2.1094}$	$\mathbf{-1.2949}$
3	-1.125	-0.25	-0.6875	$\mathbf{-1.2949}$	$\mathbf{2.1094}$	$\mathbf{-0.6875}$
4	-1.125	-0.6875	-0.9063	$\mathbf{-1.2949}$	$\mathbf{-0.6875}$	$\mathbf{0.75358}$

Then $p_4 = -0.9063$

Now, let us use the above method to solve the following nonlinear programming problem.

$$\begin{array}{ll} \text{Max (or Min)} & f(x) \\ \text{s.t.} & a \le x \le b \end{array}$$

Bisection steps

Step 1: calculate the middle of the interval [a,b], say, $c = \frac{a+b}{2}$. If $|f'(c)| \le \epsilon, \epsilon > 0$ (very small value) this means that we reach the root, $x^* = c$. But if $|f'(c)| \nleq \epsilon$ we go to step 2.

Step2:

Case 1: if $f'(a)f'(c) < 0$ this means that the approximate root will be in the interval [a,c] and we reject the interval [b,c]. Then repeat step 1 until we reach the approximate root.

Case 2: if $f'(b)f'(c) < 0$ this means that the approximate root will be in the interval [c,b] and we reject the interval [a,c]. Then repeat step 1 until we reach the approximate root.

Ex: Use the bisection method to find the optimum solution for the following nonlinear optimization problem

$$\text{Min} \quad f(x) = \frac{1}{3}x^3 - 2x,$$
$$\text{s.t} \quad 1.4 \le x \le 1.42; \ \epsilon = 10^{-2}$$

Let $f(x) = \frac{1}{3}x^3 - 2x \Rightarrow f'(x) = x^2 - 2$

n	a	b	c	$f'(a)$	$f'(b)$	$f'(c)$	$\lvert f'(c) \rvert$
1	1.4	1.42	1.41	-0.04	0.0164	-0.0119	$\nleq \epsilon$
2	1.41	1.42	1.415	-0.0119	0.0164	0.00223	$< \epsilon$

Then the optimum solution is $x^* = c = 1.415$ and $f_{min} = -1.88561$

Ex: Use the bisection method to find the optimum solution for the following nonlinear optimization problem

$$\text{Min} \quad f(x) = x^3 - 3x^2 + 5,$$
$$\text{s.t} \quad 1 \le x \le 5; \ \epsilon = 10^{-2}$$

Let $f(x) = x^3 - 3x^2 + 5 \Rightarrow f'(x) = 3x^2 - 6x$

n	a	b	c	$f'(a)$	$f'(b)$	$f'(c)$	$\lvert f'(c) \rvert$
1	1	5	3	-3	45	9	$\nleq \epsilon$
2	1	3	2	-3	9	0	$< \epsilon$

Then the optimum solution is $x^* = c = 2$ and $f_{min} = 1$

Ex: Use the bisection method to find the optimum solution for the following nonlinear optimization problem.

$$\text{Min} \quad f(x) = x + \frac{1}{x},$$
$$\text{s.t} \quad 0.5 \le x \le 1.5; \ \epsilon = 10^{-2}$$

Let $f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - \frac{1}{x^2}$

n	a	b	c	$f'(a)$	$f'(b)$	$f'(c)$	$\lvert f'(c) \rvert$
1	0.5	1.5	1	-3	5/9	0	$< \epsilon$

Then the optimum solution is $x^* = c = 1$ and $f_{min} = 2$

Ex: Use the bisection method to find the optimum solution for the following nonlinear optimization problem.

$$\text{Max} \quad f(x) = x^4 - 2x^3 - 4x^2 + 4x + 4,$$
$$\text{s.t} \quad 0 \le x \le 1; \; \epsilon = 10^{-2}$$

Let $f(x) = x^4 - 2x^3 - 4x^2 + 4x + 4 \Rightarrow f'(x) = 4x^3 - 6x^2 - 8x + 4$

n	a	b	c	f′(a)	f′(b)	f′(c)	|f′(c)|
1	0	1	0.5	4	−6	−1	$\nleq \epsilon$
2	0	0.5	0.25	4	−1	1.6875	$\nleq \epsilon$
3	0.25	0.5	0.375	1.6875	−1	0.36719	$\nleq \epsilon$
4	0.375	0.5	0.4375	0.36719	−1	−0.31348	$\nleq \epsilon$
5	0.375	0.4375	0.4063	0.36719	−0.31348	0.02795	$\nleq \epsilon$
6	0.4063	0.4375	0.4219	0.02795	−0.31348	−0.14281	$\nleq \epsilon$
7	0.4063	0.4219	0.4141	0.02795	−0.14281	−0.05764	$\nleq \epsilon$
8	0.4063	0.4141	0.4102	0.02795	−0.05764	−0.0151	$\nleq \epsilon$
9	0.4063	0.4102	0.4083	0.02795	−0.0151	0.0056156	$< \epsilon$

Then the optimum solution is $x^* = c = 0.4083$ and $f_{max} = 4.85802$

Ex: Use the bisection method to find the optimum solution for the following nonlinear optimization problem.

$$\text{Min} \quad f(x) = x^2 + \frac{1}{x},$$
$$\text{s.t} \quad 0.5 \le x \le 1.5; \; \epsilon = 10^{-2}$$

Let $f(x) = x^2 + \frac{1}{x} \Rightarrow f'(x) = 2x - \frac{1}{x^2}$

n	a	b	c	f′(a)	f′(b)	f′(c)	|f′(c)|
1	0.5	1.5	1	−3	2.5556	1	$\nleq \epsilon$
2	0.5	1	0.75	−3	1	−0.27778	$\nleq \epsilon$
3	0.75	1	0.875	−0.27778	1	0.44388	$\nleq \epsilon$
4	0.75	0.875	0.8125	−0.27778	0.44388	0.11021	$\nleq \epsilon$
5	0.75	0.8125	0.7813	−0.27778	0.11021	−0.075590	$\nleq \epsilon$
6	0.7813	0.8125	0.7969	−0.075590	0.11021	0.01912	$\nleq \epsilon$
7	0.7813	0.7969	0.7891	−0.075590	0.01912	−0.02776	$\nleq \epsilon$
8	0.7891	0.7969	0.793	−0.02776	0.01912	−0.0042069	$< \epsilon$

Then the optimum solution is $x^* = c = 0.793$ and $f_{min} = 1.88988$.

Newton – Raphson

Suppose the function $f: [a, b] \to \mathbb{R}$ is defined and continuous on its domain. In addition, its derivative existed in the open interval (a, b).

Method steps for the root of a function:
Step 1: Guess the initial root x_0.
Step 2: use the following recursive relation to get the other roots

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; n = 0,1,2, \dots, f'(x_n) \neq 0$$

Step 3: If $|f(x_{n+1})| \leq \epsilon$ then $x^* = x_{n+1}$ is the best approximate root for the function.

Method steps for the optimum solution of a nonlinear optimization problem:
Suppose the following problem:

$$\begin{array}{ll} \text{Max (or Min)} & f(x) \\ \text{s. t.} & a \leq x \leq b \end{array}$$

Step 1: Guess the initial optimum solution at x_0.
Step 2: use the following recursive relation to get the other roots

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}; n = 0,1,2, \dots, f''(x_n) \neq 0$$

Step 3: If $|f'(x_{n+1})| \leq \epsilon$ then $x^* = x_{n+1}$ is the best approximate optimum solution.

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}; n = 0,1,2, \dots, f''(x_n) \neq 0$$

Ex: Find the approximate root for the function $f(x) = e^{-x} - \sin\left(\frac{x\pi}{2}\right)$. Assume that $x_o = 1$ with efficiency degree $\epsilon = 10^{-4}$.

$$\because f(x) = e^{-x} - \sin\left(\frac{x\pi}{2}\right) \quad \Rightarrow f'(x) = -e^{-x} - \frac{\pi}{2}\cos\left(\frac{x\pi}{2}\right)$$

$$\therefore f(x_n) = e^{-x_n} - \sin\left(\frac{x_n\pi}{2}\right) \Rightarrow f'(x_n) = -e^{-x_n} - \frac{\pi}{2}\cos\left(\frac{x_n\pi}{2}\right)$$

Substituting in the recursive relation we get,

$$x_{n+1} = x_n - \frac{e^{-x_n} - \sin\left(\frac{x_n\pi}{2}\right)}{-e^{-x_n} - \frac{\pi}{2}\cos\left(\frac{x_n\pi}{2}\right)}; n = 0,1,2, \dots, f'(x_n) \neq 0$$

$$x_1 = x_o - \frac{e^{-x_o} - \sin\left(\frac{x_o\pi}{2}\right)}{-e^{-x_o} - \frac{\pi}{2}\cos\left(\frac{x_o\pi}{2}\right)} = 1 - \frac{e^{-1} - \sin\left(\frac{\pi}{2}\right)}{-e^{-1} - \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)} = 1 - \frac{e^{-1} - 1}{-e^{-1} - 0} = 2 - e$$
$$\cong -0.7183, |f(x_1)| \nleq \epsilon$$

$$x_2 = x_1 - \frac{e^{-x_1} - \sin\left(\frac{x_1\pi}{2}\right)}{-e^{-x_1} - \frac{\pi}{2}\cos\left(\frac{x_1\pi}{2}\right)} = -0.7183 - \frac{e^{-0.7183} - \sin\left(\frac{\pi}{2} \times -0.7183 \times \frac{180}{\pi}\right)}{-e^{0.7183} - \frac{\pi}{2}\cos\left(\frac{\pi}{2} \times -0.7183 \times \frac{180}{\pi}\right)}$$
$$\cong 0.3666, |f(x_2)| \nleq \epsilon$$

$$x_3 = 0.3666 - \frac{e^{-0.3666} - \sin\left(\frac{\pi}{2} \times 0.3666 \times \frac{180}{\pi}\right)}{-e^{-0.3666} - \frac{\pi}{2}\cos\left(\frac{\pi}{2} \times 0.3666 \times \frac{180}{\pi}\right)} \cong 0.4405, |f(x_3)| \nleq \epsilon$$

$$x_4 = 0.4405 - \frac{e^{-0.4405} - \sin\left(\frac{\pi}{2} \times 0.4405 \times \frac{180}{\pi}\right)}{-e^{-0.4405} - \frac{\pi}{2}\cos\left(\frac{\pi}{2} \times 0.4405 \times \frac{180}{\pi}\right)} \cong 0.4436,$$
$$|f(x_4)| \cong 0.000048866 < \epsilon$$

Then the approximate root is $x^* = x_4 = 0.4436$.

Ex: Find the approximate root for the function $f(x) = x^3 - \sqrt{x} - 1$. Assume that $x_o = 1.5$ with efficiency degree $\epsilon = 10^{-4}$.

$$\because f(x) = x^3 - \sqrt{x} - 1 \quad \Rightarrow f'(x) = 3x^2 - \frac{1}{2\sqrt{x}}$$

Substituting in the recursive relation we get,

$$x_{n+1} = x_n - \frac{x_n^3 - \sqrt{x_n} - 1}{3x_n^2 - \frac{1}{2\sqrt{x_n}}}; n = 0,1,2, \dots, f'(x_n) \neq 0$$

$$x_1 = 1.5 - \frac{(1.5)^3 - \sqrt{1.5} - 1}{3(1.5)^2 - \frac{1}{2\sqrt{1.5}}} \cong 1.3186, \qquad |f(x_1)| \nleq \epsilon$$

$$x_2 = 1.3186 - \frac{(1.3186)^3 - \sqrt{1.3186} - 1}{3(1.3186)^2 - \frac{1}{2\sqrt{1.3186}}} \cong 1.2884, \qquad |f(x_2)| \cong 0.003634 \nleq \epsilon$$

$$x_3 = 1.2884 - \frac{(1.2884)^3 - \sqrt{1.2884} - 1}{3(1.2884)^2 - \frac{1}{2\sqrt{1.2884}}} \cong 1.2876, \qquad |f(x_3)| \cong 0.000005 < \epsilon$$

Then the approximate root is $x^* = x_3 = 1.2876$.

Ex: Find an approximate value for $\sqrt{2}$. Assume that $x_o = 1.2$ with efficiency degree $\epsilon = 10^{-4}$.

Let $x = \sqrt{2} \Rightarrow x^2 = 2 \Rightarrow f(x) = x^2 - 2, f'(x) = 2x$

Then,

$$x_{n+1} = \left(\frac{x_n}{2} + \frac{1}{x_n}\right); n = 0,1,2,\ldots$$

$$x_1 = \left(1.2 + \frac{2}{1.2}\right) \cong 1.4333, \qquad |f(x_1)| \nleq \epsilon$$

$$x_2 = \left(1.4333 + \frac{2}{1.4333}\right) \cong 1.4143, \qquad |f(x_2)| \cong 0.003634 \nleq \epsilon$$

$$x_2 = \left(1.4143 + \frac{2}{1.4143}\right) \cong 1.4142, \qquad |f(x_2)| \cong 0.00004 < \epsilon$$

So, the approximate value for $\sqrt{2}$ is $x_2 = 1.4142$.

Ex: Find the approximate root for the equation $x^3 - 2x - 5 = 0$. Assume that $x_o = 2$ with efficiency degree $\epsilon = 10^{-4}$.

$\because f(x) = x^3 - 2x - 5 \qquad \Rightarrow f'(x) = 3x^2 - 2$

Substituting in the recursive relation we get,

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} = \frac{2x_n^3 + 5}{3x_n^2 - 2}; n = 0,1,2, \ldots$$

$$x_1 = \frac{2(2)^3 + 5}{3(2)^2 - 2} = 2.1, \qquad |f(x_1)| \nleq \epsilon$$

$$x_2 = \frac{2(2.1)^3 + 5}{3(2.1)^2 - 2} = 2.094568121, \qquad |f(x_2)| \nleq \epsilon$$

$$x_3 = \frac{2(2.094568121)^3 + 5}{3(2.094568121)^2 - 2} = 2.094551482, \qquad |f(x_3)| = 6 \times 10^{-9} < \epsilon$$

Then the approximate root is $x^* = x_3 = 2.094551482$.

Ex: Find the approximate root for the equation $e^{2x} = x + 5$. Assume that $x_o = 0.96$ with efficiency degree $\epsilon = 10^{-3}$.

$\because f(x) = e^{2x} - x - 5 \qquad \Rightarrow f'(x) = 2e^{2x} - 1$

Substituting in the recursive relation we get,

$$x_{n+1} = x_n - \frac{e^{2x_n} - x_n - 5}{2e^{2x_n} - 1} = \frac{(2x_n - 1)e_n^{2x} + 1}{2e^{2x_n} - 1}; n = 0,1,2, \ldots$$

$$x_1 = \frac{(2(0.96) - 1)e_n^{2(0.96)} + 1}{2e^{2(0.96)} - 1} \cong 0.9710, \qquad |f(x_1)| \nleq \epsilon$$

$$x_2 = \frac{(2(0.9710) - 1)e_n^{2(0.9710)} + 1}{2e^{2(0.9710)} - 1} \cong 0.9709, \qquad |f(x_2)| \cong 3.88 \times 10^{-4} < \epsilon$$

Then the approximate root is $x^* = x_2 = 0.9709$.

Ex: Find the approximate optimum solution for the following nonlinear optimization problem.

$$\begin{aligned} &\text{Min} \quad S(x) = x^2 + \ln^2(x)\,, \\ &\text{s.t} \quad 0.5 \le x \le 1; x_o = 0.65, \epsilon = 10^{-8} \end{aligned}$$

$$\because S(x) = x^2 + \ln^2(x) \Rightarrow S'(x)$$
$$= 2x + \frac{2}{x}\ln(x) = \frac{2}{x}(x^2 + \ln(x)), S''(x) = 2\left(1 + \frac{1}{x^2} - \frac{2}{x^2}\ln(x)\right)$$

Substituting in the recursive relation we get,

$$x_{n+1} = x_n - \frac{\frac{2}{x_n}(x_n^2 + \ln(x_n))}{2\left(1 + \frac{1}{x_n^2} - \frac{2}{x_n^2}\ln(x_n)\right)} = \frac{x_n - 2x_n\ln(x_n)}{1 + x_n^2 - \ln(x_n)}; n = 0,1,2, \ldots,$$

$$x_1 = \frac{0.65 - 2(0.65)\ln(0.65)}{1 + (0.65)^2 - \ln(0.65)} = 0.65290506, \qquad |S'(x_1)| \cong 1.18 \times 10^{-4} \nleq \epsilon$$

$$x_2 = \frac{0.65290506 - 2(0.65290506)\ln(0.65290506)}{1 + (0.65290506)^2 - \ln(0.65290506)} = 0.65291864,$$
$$|S'(x_2)| \cong 3.64 \times 10^{-9} < \epsilon$$

Then the optimum solution is $x^* = 0.65291864$ and hence, $S_{min} \cong 0.60803679$.

Ex: Find the approximate optimum solution for the following nonlinear optimization problem.

$$\begin{aligned} &\text{Min} \quad f(x) = x + 1/x\,, \\ &\text{s.t} \quad 0.5 \le x \le 1.5; x_o = 0.6, \epsilon = 10^{-9} \end{aligned}$$

$$\because f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - \frac{1}{x^2}, f''(x) = \frac{2}{x^3}$$

Substituting in the recursive relation we get,

$$x_{n+1} = x_n - \frac{1 - \frac{1}{x_n^2}}{\frac{2}{x_n^3}} = \frac{-x_n(x_n^2 - 3)}{2}; n = 0,1,2, \ldots,$$

$$x_1 = \frac{-0.6[(0.6)^2 - 3]}{2} = 0.792, \qquad |f'(x_1)| \cong 0.5942250791 \nleq \epsilon$$

$$x_2 = \frac{-0.792[(-0.792)^2 - 3]}{2} = 0.939603456, \qquad |f'(x_2)| \cong 0.1326892766 \nleq \epsilon$$

$$x_3 = \frac{-0.939603456[(0.939603456)^2 - 3]}{2} = 0.9946385417, \qquad |f'(x_3)| \cong 0.5942250791 \nleq \epsilon$$

$$x_4 = \frac{-0.9946385417[(-0.9946385417)^2 - 3]}{2} = 0.9999569592, \qquad |f'(x_4)| \cong 8.61 \times 10^{-5} \nleq \epsilon$$

$$x_5 = \frac{-0.9999569592[(0.9999569592)^2 - 3]}{2} = 0.9999999972, \qquad |f'(x_5)| \cong 5.6 \times 10^{-9} \nleq \epsilon$$

$$x_6 = \frac{-0.9999999972[(-0.9999999972)^2 - 3]}{2} = 1, \qquad |f'(x_6)| = 0 < \epsilon$$

Then the optimum solution is $x^* = 1$ and hence, $f_{min} = 2$.