College of Science
Department of Statistics & OR

OR 122
Introduction to Operations Research

Chapter 7:

Decision Theory

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Structure of Decision-Making Problem

1) The decision maker: who is charged with the responsibility for making decision-selection of one from a set of possible courses of action.

2) Acts: are the alternative courses of action or strategies, that are available to decision maker.

⇒ The decision maker has a control over choice of these acts.

3) States of nature (events): Determine the level of success for a given act.

⇒ The decision-maker has no control on them.

4) Uncertainty: It is indicated in terms of probabilities assigned to events.

Structure of Decision-Making Problem

5) Types of information:

a) Perfect information on problem.

$\Rightarrow$ In this case, we have decision making under certainty.

b) Partial or imperfect information.

$\Rightarrow$ In this case, we have two types of decision-making situations that are:

i) Decision under risk.

ii) Decision under uncertainty.

Structure of Decision-Making Problem

6) Payoff table: it takes the following form

	States of nature			
Alternatives	S_1	S_2	$\cdots$	S_n
	P_1	P_2	$\cdots$	P_n
A_1	r_{11}	r_{12}	$\cdots$	r_{1n}
A_2	r_{21}	r_{22}	$\cdots$	r_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_m	r_{m1}	r_{m2}	$\cdots$	r_{mn}

Where $P_i; i = 1,2, \dots, n$, is the probability to each state of nature, and $r_{ji};\ i = , 1,2, \dots, n, j = 1,2, \dots, m$ is the payoff due to selecting the alternative j.

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Example 1

A manger has three devices that are defect A, B, and C. He has only three technicians to fix these devices. Experiences show that the following table presents the time taken to fix these devices by the three technicians. Suppose the manger will assign only one for each device. So, what is the best choice for him to get a minimum time of fixing the devices:

	Devices		
Technicians	A	B	C
J	3	7	4
G	4	6	6
M	3	8	5

Solution

This is a problem of perfect information. So, we have the following alternatives:

$a_1: J(A), G(B), M(C) \Rightarrow t_1 = 3 + 6 + 5 = 14$

$a_2: J(A), G(C), M(B) \Rightarrow t_2 = 3 + 6 + 8 = 17$

$a_3: J(B), G(A), M(C) \Rightarrow t_3 = 7 + 4 + 5 = 16$

$a_4: J(B), G(C), M(A) \Rightarrow t_4 = 7 + 6 + 3 = 16$

$a_5: J(C), G(B), M(A) \Rightarrow t_5 = 4 + 6 + 3 = 13$

$a_6: J(C), G(A), M(B) \Rightarrow t_6 = 4 + 4 + 8 = 16$

So, the best alternative is to assign $J \to C, G \to B, M \to A$ with minimum time $t_5 = 13$.

Decisions under risk:

In this situation, the decision-maker possesses some measures to identify the best alternative.

1) The expected payoff criterion (E).

2) The expected opportunity loss criterion (EOL), sometimes it is called regret criterion.

3) The criterion of most likelihood.

Example 2

An investor wants to invest an amount of money during a year. He has the following alternatives and the states of nature with their probability and the profit for each choice.

	State of nature		
	Inflation $p_1 = 0.2$	Recession $p_2 = 0.3$	Growth $p_3 = 0.5$
Furniture (a_1)	7	8	12
Stock market (a_2)	-2	10	25
Cars (a_3)	6.5	8.5	16.5

What is the best alternative according to the expected payoff criterion.

Find the expectation, then take the max value if the question about profit or take the min value if the question about cost.

Solution

$$E(a_1) = 7(0.2) + 8(0.3) + 12(0.5) = 9.8$$

$$E(a_2) = -2(0.2) + 10(0.3) + 25(0.5) = 15.1$$

$$E(a_3) = 6.5(0.2) + 8.5(0.3) + 16.5(0.5) = 12.1$$

So, the best alternative for the investor is to invest the money in stock market as $E(a_2) = 15.1$ is the greatest value.

Example 3

Solve the previous example using the other two criteria: the expected opportunity loss criterion and the criterion of most likelihood.

Solution:

Expected opportunity loss criterion: Construct the regret table. Then, find the expectation, then take the min value if the question about profit, or cost.

Regret table ⇒

	State of nature		
	S_1 (0.2)	S_2 (0.3)	S_3 (0.5)
a_1	7-7	10-8	25-12
a_2	7-(-2)	10-10	25-25
a_3	7-6.5	10-8.5	25-16.5

⇒

	State of nature		
	S_1 (0.2)	S_2 (0.3)	S_3 (0.5)
a_1	0	2	13
a_2	9	0	0
a_3	0.5	1.5	8.5

Solution

$$EOL(a_1) = 0(0.2) + 2(0.3) + 13(0.5) = 7.1$$

$$EOL(a_2) = 9(0.2) + 0(0.3) + 0(0.5) = 1.8$$

$$EOL(a_3) = 0.5(0.2) + 1.5(0.3) + 8.5(0.5) = 4.8$$

According to this measure we have the best alternative a_2 as it has the minimum EOL.

Solution

Using the third criterion (most likelihood):

Consider the highest probability column, then take the max for the profit table or take the min for the cost table.

	S_3 (0.5)
a_1	12
a_2	25
a_3	16.5

The highest probability is 0.5 for the growth state, from the opposite table, we have a_2 with the highest payoff. Then, a_2 is the best alternative.

Example 4

For the following cost table find the best alternative?

	State of nature			
	S_1 (0.3)	S_2 (0.1)	S_3 (0.4)	S_4 (0.2)
a_1	8	9	5	12
a_2	10	12	6	12
a_3	17	5	8	15

Solution

Expected payoff criterion:

$$E(a_1) = 8(0.3) + 9(0.1) + 5(0.4) + 12(0.2) = 7.70$$

$$E(a_2) = 10(0.3) + 12(0.1) + 6(0.4) + 12(0.2) = 9$$

$$E(a_3) = 17(0.3) + 5(0.1) + 8(0.4) + 15(0.2) = 11.80$$

According to the expected payoff criterion a_1 is the best alternative.

Solution

Expected opportunity loss criterion:

Regret table $\Rightarrow$

	State of nature			
	S_1 (0.3)	S_2 (0.1)	S_3 (0.4)	S_4 (0.2)
a_1	8-8=0	9-5=4	5-5=0	12-12=0
a_2	10-8=2	12-5=7	6-5=1	12-12=0
a_3	17-8=9	5-5=0	8-5=3	15-12=3

$$EOL(a_1) = 0(0.3) + 4(0.1) + 0(0.4) + 0(0.2) = 0.4$$

$$EOL(a_2) = 2(0.3) + 7(0.1) + 1(0.4) + 0(0.2) = 1.7$$

$$EOL(a_3) = 9(0.3) + 0(0.1) + 3(0.4) + 3(0.2) = 4.50$$

According to the EOL we have a_1 is the best alternative.

Solution

Using the third criterion (most likelihood):

	S_3 **(0.4)**
a_1	5
a_2	6
a_3	8

We have a_1 is the best alternative, because it has the minimum cost.

Suppose the following matrix, find the best alternative by using the most likelihood criterion.

	S_1 (0.4)	S_2 (0.4)	S_3 (0.2)
a_1	1	3	2
a_2	1	4	3
a_3	2	5	7

We need to find the average for the first and second columns then take the max for the profit table or take min for the cost table.

Answer:

The best alternative is a_1 (if the matrix represents the cost),

and the best alternative is a_3 (if the matrix represents the profit).

Decision under uncertainty

- The decision-maker in this types does not have any information about the state of nature (events). Instead, he has only these states.
- Examples are the new phenomena such as increasing prices of Gold, Oli, Gas, …, etc.
- For this type we have 5 criteria that are:

1) Laplace criterion: the probability of all states are equal.
2) Pessimistic criterion:
   - For profit we use maximin.
   - For cost we use minimax.
3) Optimistic criterion:
   - For profit we use maximax.
   - For cost we use minimin.
4) Hurwicz criterion.
5) Savage criterion or regret criterion: we apply minimax on the regret table.

Hurwicz Criterion

He gives a probability of optimistic (α in which $0 \leq \alpha \leq 1$) and then evaluate the expected value for each alternative as follows:

$$V(a_i) = \alpha * \binom{\text{maximum profit}}{\text{OR minimum cost}} + (1 - \alpha)\binom{\text{minimum profit}}{\text{OR maximum cost}}$$

The following examples show how can we use these criteria of decision under uncertainty.

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Example 5

What is the best alternative for the following cost table.

	S_1	S_2	S_3	S_4
a_1	5	8	3	1
a_2	7	4	5	2
a_3	3	6	6	4

Solution

1) Using Laplace:

Let $P(S_1) = P(S_2) = P(S_3) = P(S_4)$

$\Rightarrow P(S_i) = \frac{1}{4}; \quad i = 1,2,3,4.$

$E(a_1) = \frac{1}{4}(5 + 8 + 3 + 1) = \frac{17}{4} = 4.25$

$E(a_2) = \frac{1}{4}(7 + 4 + 5 + 2) = \frac{18}{4} = 4.50$

$E(a_3) = \frac{1}{4}(3 + 6 + 6 + 4) = \frac{19}{4} = 4.75$

According to Laplace a_1 is the optimum alternative.

Solution

2) Using the pessimistic criterion:

First find the maximum cost in each alternative, then chose the smallest one.

	Worst (cost)
a_1	8
a_2	7
a_3	6

This means that minimax $= 6$. So, a_3 is the optimum alternative.

Solution

3) Using the optimistic criterion:

First find the minimum cost in each alternative, then chose the smallest one.

	Low (cost)
a_1	1
a_2	2
a_3	3

This means that minimin $= 1$. So, a_1 is the optimum alternative.

4) Using Hurwicz with $\alpha = 0.4 \Rightarrow 1 - \alpha = 0.6$

$\mathrm{V}(a_1) = 1(0.4) + 8(0.6) = 5.2$

$\mathrm{V}(a_2) = 2(0.4) + 7(0.6) = 5$

$\mathrm{V}(a_3) = 3(0.4) + 6(0.6) = 4.8$

So, a_3 is the optimum alternative.

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5) Using Savage criterion:

We construct the regret table first, then take minimax.

	S_1	S_2	S_3	S_4
a_1	5-3	8-4	3-3	1-1
a_2	7-3	4-4	5-3	2-1
a_3	3-3	6-4	6-3	4-1

$\Rightarrow$

	S_1	S_2	S_3	S_4
a_1	2	4	0	0
a_2	4	0	2	1
a_3	0	2	3	3

$\Rightarrow$

	Max
a_1	4
a_2	4
a_3	3

So, a_3 is the optimum alternative.

Example 6

What is the optimum alternative for the following profit table.

	S_1	S_2	S_3
a_1	7	2	-1
a_2	3	6	2
a_3	0	3	8

Solution

1) Using Laplace:

Let $P(S_1) = P(S_2) = P(S_3)$

$\Rightarrow P(S_i) = \frac{1}{3}; \quad i = 1,2,3.$

$E(a_1) = \frac{1}{3}(7 + 2 - 1) = \frac{8}{3} \cong 2.7$

$E(a_2) = \frac{1}{3}(3 + 6 + 2) = \frac{11}{3} \cong 3.7$

$E(a_3) = \frac{1}{3}(0 + 3 + 8) = \frac{11}{3} \cong 3.7$

Both a_2 and a_3 are optimum alternatives.

Solution

2) Using the pessimistic criterion:

First find the minimum profit in each alternative, then chose the largest one.

	Worst (profit)
a_1	-1
a_2	2
a_3	0

This means that maximin $= 2$. So, a_2 is the optimum alternative.

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Solution

3) Using the optimistic criterion:

First find the maximum profit in each alternative, then chose the largest one.

	High (profit)
a_1	7
a_2	6
a_3	8

This means that maximax $= 8$. So, a_3 is the optimum alternative.

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4) Using Hurwicz with $\alpha = 0.4 \Rightarrow 1 - \alpha = 0.6$

$\text{V}(a_1) = 7(0.4) + (-1)(0.6) = 2.2$

$\text{V}(a_2) = 6(0.4) + 2(0.6) = 3.6$

$\text{V}(a_3) = 8(0.4) + (0)(0.6) = 3.2$

So, a_2 is the optimum alternative.

5) Using Savage criterion:

We construct the regret table first, then take minimax.

	S_1	S_2	S_3
a_1	7-7	6-2	8-(-1)
a_2	7-3	6-6	8-2
a_3	7-0	6-3	8-8

$\Rightarrow$

	S_1	S_2	S_3
a_1	0	4	9
a_2	4	0	6
a_3	7	3	0

$\Rightarrow$

	Max
a_1	9
a_2	6
a_3	7

This means that, a_2 is the optimum alternative.

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Example 7

Hurwicz Criterion:

Consider the following profit table:

	S_1	S_2	S_3
a_1	3	6	-1
a_2	8	5	4
a_3	-4	7	12

What is the probability of optimism ($\alpha = ??$) that makes a_1 the optimum alternative?

$\alpha = ??$ $\quad 0 \le \alpha \le 1 \Rightarrow (1)$

$V(a_1) = 6(\alpha) + (-1)(1-\alpha) = 7\alpha - 1$

$V(a_2) = 8(\alpha) + 4(1-\alpha) = 4\alpha + 4$

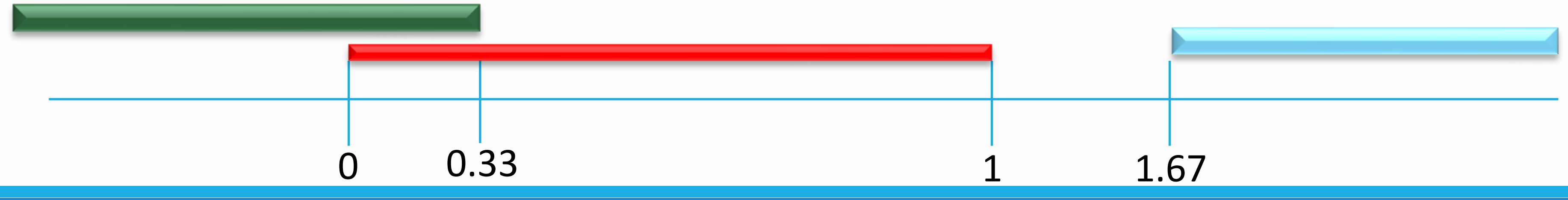
$V(a_3) = 12(\alpha) + (-4)(1-\alpha) = 16\,\alpha - 4$

$a^* = a_1$

$\Rightarrow V[a_1] > V[a_2] \Rightarrow 7\alpha - 1 > 4\alpha + 4 \Rightarrow 3\alpha > 5 \Rightarrow \alpha > 1.67 \;\Rightarrow (2)$

and, $V[a_1] > V[a_3] \Rightarrow 7\alpha - 1 > 16\,\alpha - 4 \Rightarrow 9\alpha < 3 \Rightarrow \alpha < 0.33 \;\Rightarrow (3)$

There is no value for α that makes a_1 optimum alternative. There is no α that satisfy (1), (2), and (3) .



What is the probability of optimism that makes a_3 the optimum alternative?

$\alpha = ??$ $\quad\quad 0 \leq \alpha \leq 1 \Rightarrow (1)$

$V(a_1) = 6(\alpha) + (-1)(1-\alpha) = 7\alpha - 1$

$V(a_2) = 8(\alpha) + 4(1-\alpha) = 4\alpha + 4$

$V(a_3) = 12(\alpha) + (-4)(1-\alpha) = 16\,\alpha - 4$

$a^* = a_3$

$\Rightarrow V[a_3] > V[a_1] \Rightarrow 16\,\alpha - 4 > 7\alpha - 1 \Rightarrow 9\alpha > 3 \Rightarrow \alpha > 0.33 \;\Rightarrow (2)$

and, $V[a_3] > V[a_2] \Rightarrow 16\,\alpha - 4 > 4\alpha + 4 \Rightarrow 12\alpha > 8 \Rightarrow \alpha > 0.67 \;\Rightarrow (3)$

For all $0.67 < \alpha \leq 1 \Rightarrow a^* = a_3$

Example 8

Consider the following cost table:

	S_1	S_2	S_3	S_4
a_1	8	9	5	12
a_2	10	12	6	12
a_3	17	5	8	15

What is the probability of optimism that makes a_1 the optimum alternative?

$\alpha = ??$ $0 \leq \alpha \leq 1$

$V(a_1) = 5(\alpha) + 12(1 - \alpha) = -7\alpha + 12$

$V(a_2) = 6(\alpha) + 12(1 - \alpha) = -6\alpha + 12$

$V(a_3) = 5(\alpha) + 17(1 - \alpha) = -12\alpha + 17$

$a^* = a_1$

$\Rightarrow V[a_1] < V[a_2] \Rightarrow -7\alpha + 12 < -6\alpha + 12 \Rightarrow -\alpha < 0 \Rightarrow \alpha > 0$

and, $V[a_1] < V[a_3] \Rightarrow -7\alpha + 12 < -12\alpha + 17 \Rightarrow 5\alpha < 5 \Rightarrow \alpha < 1$

For all $0 < \alpha < 1 \Rightarrow a^* = a_1$

What is the probability of optimism that makes a_2 the optimum alternative?

$\alpha = ??$ $\quad 0 \leq \alpha \leq 1$

$V(a_1) = 5(\alpha) + 12(1-\alpha) = -7\alpha + 12$

$V(a_2) = 6(\alpha) + 12(1-\alpha) = -6\alpha + 12$

$V(a_3) = 5(\alpha) + 17(1-\alpha) = -12\alpha + 17$

$a^* = a_2$

$\Rightarrow V[a_2] < V[a_1] \Rightarrow -6\,\alpha + 12 < -7\alpha + 12 \Rightarrow \alpha < 0$

and, $V[a_2] < V[a_3] \Rightarrow -6\alpha + 12 < -12\alpha + 17 \Rightarrow 6\alpha < 5 \Rightarrow \alpha < 0.83$

There is no solution that makes a_2 optimum solution.