College of Science
Department of Statistics & OR

OR 122
Introduction to Operations Research

# Chapter 6:

## Assignment Model

**Note:** These class notes were originally prepared by Prof. Sameh Asker, Dr. Wael Al Hajailan, Dr. Adel Alrasheedi, and have been subsequently revised and improved by: Dr. Razan Alsehibani, Dr. Kholood Alyazidi and Alanoud Alzughaibi.

Definition:

It is a scheduling model used for assigning individuals jobs to individual processing components on a one-to-one basis with the goal of minimizing the total costs or time for accomplishing all the jobs.

## The mathematical description:

Let $x_{ij}$ denotes the assignment of the $i^{th}$ worker to the $j^{th}$ project.

$$\Rightarrow \quad x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to } j \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$s.t. \begin{cases} \sum_{j=1}^{n} x_{ij} = 1 & \text{(workers condition) for all } i \\ \sum_{i=1}^{n} x_{ij} = 1 & \text{(jobs condition) for all } ij \end{cases}$$

| Activity (projects) | | | | | |
|---|---|---|---|---|---|
| | $P_1$ | $P_2$ | $\cdots$ | $P_n$ | Capacity |
| $W_1$ | $c_{11}$ | $c_{12}$ | $\cdots$ | $c_{1n}$ | 1 |
| $W_2$ | $c_{21}$ | $c_{22}$ | $\cdots$ | $c_{2n}$ | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ |
| $W_n$ | $c_{n1}$ | $c_{n2}$ | $\cdots$ | $c_{nn}$ | 1 |
| | 1 | 1 | $\cdots$ | 1 | |

Note: Do not forget to make the problem balanced

# Hungarian Method

It is used to solve assignment problem

Step 1: Subtract the smallest number in each row from every number in that row and then subtract the smallest number in each column from every number in the column.

Step2: Draw the minimum number of vertical and horizontal straight lines necessary to cover all zero elements in the table.

Optimality test: if the number of lines equals either the number of rows or the number of columns in the table then the solution is optimum, and we produce to the next stop. Otherwise, we go to step 3.

Step 3: Subtract the smallest number not covered by a line from every uncovered number. Add the same number to any number lying at the intersection of any two line. Return to Step 2 and continue until an optimum assignment is reached.

Step 4: Given the optimum solution, make the project assignment as explained in the following solved example.

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# Example 1

Consider problem of assigning five operators to five Machines. The assignment costs are given below.

| Operators | | | | | | |
|---|---|---|---|---|---|---|
| | I | II | III | IV | V | Min |
| A | 10 | 5 | 13 | 15 | 16 | 5 |
| B | 3 | 9 | 18 | 3 | 6 | 3 |
| C | 10 | 7 | 2 | 2 | 2 | 2 |
| D | 5 | 11 | 9 | 7 | 12 | 5 |
| E | 7 | 9 | 10 | 4 | 12 | 4 |

The number of rows = 5 and the number of columns = 5, it is balanced.

# Solution

The total cost is

$$\begin{aligned} Min\ z = &(10x_{11} + 5x_{12} + 13x_{13} + 15x_{14} + 16x_{15}) \\ &+ (3x_{21} + 9x_{22} + 18x_{23} + 3x_{24} + 6x_{25}) \\ &+ (10x_{31} + 7x_{32} + 2x_{33} + 2x_{34} + 2x_{35}) \\ &+ (5x_{41} + 11x_{42} + 9x_{43} + 7x_{44} + 12x_{45}) \\ &+ (7x_{51} + 9x_{52} + 10x_{53} + 4x_{54} + 12x_{55}) \end{aligned}$$

# Solution

$$
s.t.\begin{cases}
Machine\ constrints & \begin{cases}
x_{11}+x_{12}+x_{13}+x_{14}+x_{15}=1\\
x_{21}+x_{22}+x_{23}+x_{24}+x_{25}=1\\
x_{31}+x_{32}+x_{33}+x_{34}+x_{35}=1\\
x_{41}+x_{42}+x_{43}+x_{44}+x_{45}=1\\
x_{51}+x_{52}+x_{53}+x_{54}+x_{55}=1
\end{cases}\\
Operators\ constrints & \begin{cases}
x_{11}+x_{21}+x_{31}+x_{41}+x_{51}=1\\
x_{12}+x_{22}+x_{32}+x_{42}+x_{52}=1\\
x_{13}+x_{23}+x_{33}+x_{43}+x_{53}=1\\
x_{14}+x_{24}+x_{34}+x_{44}+x_{54}=1\\
x_{15}+x_{25}+x_{35}+x_{45}+x_{55}=1
\end{cases}
\end{cases}
$$

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 5 | 0 | 8 | 10 | 11 |
| B | 0 | 6 | 15 | 0 | 3 |
| C | 8 | 5 | 0 | 0 | 0 |
| D | 0 | 6 | 4 | 2 | 7 |
| E | 3 | 5 | 6 | 0 | 8 |

Minimum number of each row

$\Longrightarrow$

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 5 | 0 | 8 | 10 | 11 |
| B | 0 | 6 | 15 | 0 | 3 |
| C | 8 | 5 | 0 | 0 | 0 |
| D | 0 | 6 | 4 | 2 | 7 |
| E | 3 | 5 | 6 | 0 | 8 |

Minimum number of each column

Now, cover all zeros with minimum lines, we will start with the rows or columns that contains the largest number of zeros.

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 5 | 0 | 8 | 10 | 11 |
| B | 0 | 6 | 15 | 0 | 3 |
| C | 8 | 5 | 0 | 0 | 0 |
| D | 0 | 6 | 4 | 2 | 7 |
| E | 3 | 5 | 6 | 0 | 8 |

Min # of straight line (= 4) $\neq$ # of rows(= 5)

Then the solution is not optimal.

Now, find h= the smallest uncovered number, then subtract (h) from all uncovered cells and add (h) to all cells covered with two lines.

Then again, cover all zeros with minimum lines

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 5 | 0 | 8 | 10 | 11 |
| B | 0 | 6 | 15 | 0 | 3 |
| C | 8 | 5 | 0 | 0 | 0 |
| D | 0 | 6 | 4 | 2 | 7 |
| E | 3 | 5 | 6 | 0 | 8 |

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 5 | 0 | 5 | 10 | 8 |
| B | 0 | 6 | 12 | 0 | 0 |
| C | 11 | 8 | 0 | 3 | 0 |
| D | 0 | 6 | 1 | 2 | 4 |
| E | 3 | 5 | 3 | 0 | 5 |

Min # of straight line (= 5) = # of rows(= 5)

Then the optimum assigned is obtained.

Now, start with the assignment process, we will start with the rows or columns that contain the smallest number of zero.
When we choose a cell, we must cancel the row and the column of this cell.

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 5 | 0 | 5 | 10 | 8 |
| B | 0 | 6 | 12 | 0 | 0 |
| C | 11 | 8 | 0 | 3 | 0 |
| D | 0 | 6 | 1 | 2 | 4 |
| E | 3 | 5 | 3 | 0 | 5 |

Then $x_{12} = 1,\ x_{25} = 1,\ x_{33} = 1,\ \ x_{41} = 1,\ \ x_{54} = 1.$ and the rest equals zero.

$$\Rightarrow \text{Min cost} = 5 + 6 + 2 + 5 + 4 = 22$$

# Special Cases in Assignment Problem

1) Maximization case: In some cases, the pay off elements of the assignment problem may represent revenues or profits instead of costs so that the objective will be to maximize total revenue or profit.

$\Rightarrow$ The problem of maximization can be converted into minimization case by selecting the largest element among all elements of the profit matrix and subtracting it from all other elements in the matrix and then we proceed as in the minimization case.

# Special Cases in Assignment Problem

2) Unbalanced Assignment Problem: If number of rows are not equal to the number of columns, the assignment problem is called unbalanced assignment problem. In such cases, dummy rows and/or columns are added in the matrix with zero costs are associated with these dummies and then we apply the Hungarian method.

# Example 2

The head of the department has four subordinates and four tasks to be performed. His estimates of the times each man would take to perform each task is given below in the matrix.

How should the tasks be allocated to subordinates to minimize the total man-hours?

| | Tasks | | | |
|---|---|---|---|---|
| Subordinates | I | II | III | IV |
| A | 8 | 26 | 17 | 11 |
| B | 13 | 28 | 4 | 26 |
| C | 38 | 19 | 18 | 15 |
| D | 19 | 26 | 24 | 10 |

The number of rows = 4 and the number of columns = 4, it  is balanced.

Minimum number of each row

| | I | II | III | IV |
|---|---|---|---|---|
| A | 0 | 18 | 9 | 3 |
| B | 9 | 24 | 0 | 22 |
| C | 23 | 4 | 3 | 0 |
| D | 9 | 16 | 14 | 0 |

$\Longrightarrow$

Minimum number of each column

| | I | II | III | IV |
|---|---|---|---|---|
| A | 0 | 14 | 9 | 3 |
| B | 9 | 20 | 0 | 22 |
| C | 23 | 0 | 3 | 0 |
| D | 9 | 12 | 14 | 0 |

$\Longrightarrow$

| | I | II | III | IV |
|---|---|---|---|---|
| A | 0 | 14 | 9 | 3 |
| B | 9 | 20 | 0 | 22 |
| C | 23 | 0 | 3 | 0 |
| D | 9 | 12 | 14 | 0 |

$\Longrightarrow$ Min # of lines (=4) = # of rows (=4).

Then the optimum assignment is obtained.

So, optimum allocation is sought as follows:

| | I | II | III | IV |
|---|---|---|---|---|
| A | 0 | 14 | 9 | 3 |
| B | 9 | 20 | 0 | 22 |
| C | 23 | 0 | 3 | ~~0~~ |
| D | 9 | 12 | 14 | 0 |

Then $x_{11} = 1,\ x_{23} = 1,\ x_{32} = 1,\ \ x_{44} = 1,$ and the rest equals zero.

$\Rightarrow \text{Min time} = 8 + 4 + 19 + 10 = 41$

# Example 3

Use the Hungarian method to get the maximum profit for the following assignment matrix.

| | I | II | III |
|---|---|---|---|
| A | 5 | 4 | 7 |
| B | 6 | 7 | 3 |
| C | 8 | 11 | 2 |

The number of rows = 3 and the number of columns = 3, it is balanced.

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This problem should be converted into a minimization one.

The largest number in the matrix is =11.

| | I | II | III |
|---|---|---|---|
| A | 11-5 | 11-4 | 11-7 |
| B | 11-6 | 11-7 | 11-3 |
| C | 11-8 | 11-11 | 11-2 |

$\Longrightarrow$

| | I | II | III |
|---|---|---|---|
| A | 6 | 7 | 4 |
| B | 5 | 4 | 8 |
| C | 3 | 0 | 9 |

Minimum number of each row

| | I | II | III |
|---|---|---|---|
| A | 2 | 3 | 0 |
| B | 1 | 0 | 4 |
| C | 3 | 0 | 9 |

$\Longrightarrow$

Minimum number of each column

| | I | II | III |
|---|---|---|---|
| A | 1 | 3 | 0 |
| B | 0 | 0 | 4 |
| C | 2 | 0 | 9 |

$\Longrightarrow$

| | I | II | III |
|---|---|---|---|
| A | 1 | 3 | 0 |
| B | 0 | 0 | 4 |
| C | 2 | 0 | 9 |

$\Longrightarrow$ Min # of lines (=3) = # of rows (=3).

Then the optimum assignment is obtained.

So, optimum allocation is sought as follows:

| | I | II | III |
|---|---|---|---|
| A | 1 | 3 | 0 |
| B | 0 | ~~0~~ | 4 |
| C | 2 | 0 | ~~0~~ |

Then $x_{13} = 1,\ x_{21} = 1,\ x_{32} = 1,$ and the rest equals zero.

$\Rightarrow \text{Max profit} = 7 + 6 + 11 = 24$